



Wiener criteria for existence of large solutions of nonlinear parabolic equations with absorption in a non-cylindrical domain

Quoc-Hung Nguyen, Laurent Veron

► To cite this version:

Quoc-Hung Nguyen, Laurent Veron. Wiener criteria for existence of large solutions of nonlinear parabolic equations with absorption in a non-cylindrical domain. 2015. hal-01006258v3

HAL Id: hal-01006258

<https://hal.science/hal-01006258v3>

Preprint submitted on 26 Nov 2015

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

Wiener criteria for existence of large solutions of nonlinear parabolic equations with absorption in a non-cylindrical domain

Quoc-Hung Nguyen ^{*}
Laurent Véron [†]

Laboratoire de Mathématiques et Physique Théorique,
Université François Rabelais, Tours, FRANCE

Abstract

We obtain a necessary condition and a sufficient condition, both expressed in terms of Wiener type tests involving the parabolic $W_{q'}^{2,1}$ - capacity, where $q' = \frac{q}{q-1}$ and $q > 1$, for the existence of large solutions to equation $\partial_t u - \Delta u + u^q = 0$ in a non-cylindrical domain. We provide also a sufficient condition for the existence of such solutions to equation $\partial_t u - \Delta u + e^u - 1 = 0$. Besides, we apply our results to equation: $\partial_t u - \Delta u + a|\nabla u|^p + bu^q = 0$ for $a, b > 0$, $1 < p < 2$ and $q > 1$.

Keywords. Bessel capacities; Hausdorff capacities; parabolic boundary; Riesz potential; maximal solutions.

2010 Mathematics Subject Classification. 35K58, 28A12, 46E35.

1 Introduction

The aim of this paper is to study the problem of existence of large solutions to some nonlinear parabolic equations with superlinear absorption in an *arbitrary* bounded open set $O \subset \mathbb{R}^{N+1}$, $N \geq 2$. These are functions $u \in C^{2,1}(O)$, solutions of

$$\begin{aligned} \partial_t u - \Delta u + |u|^{q-1}u &= 0 && \text{in } O, \\ \lim_{\delta \rightarrow 0} \inf_{O \cap Q_\delta(x,t)} u &= \infty && \text{for all } (x, t) \in \partial_p O, \end{aligned} \quad (1.1)$$

with $q > 1$ and

$$\begin{aligned} \partial_t u - \Delta u + \text{sign}(u)(e^{|u|} - 1) &= 0 && \text{in } O, \\ \lim_{\delta \rightarrow 0} \inf_{O \cap Q_\delta(x,t)} u &= \infty && \text{for all } (x, t) \in \partial_p O, \end{aligned} \quad (1.2)$$

in which expressions $\partial_p O$ denotes the parabolic boundary of O , i.e. the set all points $X = (x, t) \in \partial O$ such that the intersection of the cylinder $Q_\delta(x, t) := B_\delta(x) \times (t - \delta^2, t)$ with O^c is not empty for any $\delta > 0$. By the maximal principle for parabolic equations we can assume that all solutions of (1.1) and (1.2) are positive. Henceforth we consider only positive solutions of the preceding equations.

In [23], we studied the existence and the uniqueness of solution of semilinear heat equations in a cylindrical domain,

$$\begin{aligned} \partial_t u - \Delta u + f(u) &= 0 && \text{in } \Omega \times (0, \infty), \\ u &= \infty && \text{in } \partial_p(\Omega \times (0, \infty)), \end{aligned} \quad (1.3)$$

^{*}E-mail address: Hung.Nguyen-Quoc@lmpt.univ-tours.fr, quoc-hung.nguyen@epfl.ch

[†]E-mail address: Laurent.Veron@lmpt.univ-tours.fr

where Ω is a bounded open set in \mathbb{R}^N and f a continuous nondecreasing real-valued function such that $f(0) \geq 0$ and $f(a) > 0$ for some $a > 0$. In order to obtain the existence of a maximal solution of $\partial_t u - \Delta u + f(u) = 0$ in $\Omega \times (0, \infty)$ there is need to introduce the following assumptions

$$\begin{aligned} (i) \quad & \int_a^\infty \left(\int_0^s f(\tau) d\tau \right)^{-\frac{1}{2}} ds < \infty, \\ (ii) \quad & \int_a^\infty (f(s))^{-1} ds < \infty. \end{aligned} \tag{1.4}$$

Condition (i), due to Keller and Osserman, is a necessary and sufficient for the existence of a maximal solution to

$$-\Delta u + f(u) = 0 \quad \text{in } \Omega. \tag{1.5}$$

Condition (ii) is a necessary and sufficient for the existence of a maximal solution of the differential equation

$$\varphi' + f(\varphi) = 0 \quad \text{in } (0, \infty), \tag{1.6}$$

and this solution tends to ∞ at 0. In [23], it is shown that if for any $m \in \mathbb{R}$ there exists $L = L(m) > 0$ such that

$$\text{for any } x, y \geq m \Rightarrow f(x+y) \geq f(x) + f(y) - L,$$

and if (1.5) has a large solution, then (1.3) admits a solution.

It is not always true that the maximal solution to (1.5) is a large solution. However, if f satisfies

$$\int_1^\infty s^{-2(N-1)/(N-2)} f(s) ds < \infty \quad \text{if } N \geq 3,$$

or

$$\inf \left\{ a \geq 0 : \int_0^\infty f(s) e^{-as} ds < \infty \right\} < \infty \quad \text{if } N = 2,$$

then (1.5) has a large solution for any bounded domain Ω , see [17].

When $f(u) = u^q$, $q > 1$ and $N \geq 3$, the first above condition is satisfied if and only if $q < q_c := \frac{N}{N-2}$, this is called *the sub-critical case*. When $q \geq q_c$, a necessary and sufficient condition for the existence of a large solution to

$$-\Delta u + u^q = 0 \quad \text{in } \Omega \tag{1.7}$$

is expressed in term of a Wiener-type test,

$$\int_0^1 \frac{\text{Cap}_{2,q'}(\Omega^c \cap B_r(x))}{r^{N-2}} \frac{dr}{r} = \infty \quad \text{for all } x \in \partial\Omega. \tag{1.8}$$

In the case $q = 2$ it is obtained by Dhersin and Le Gall [5], see also [13, 14], using probabilistic methods involving the Brownian snake; this method can be extended for $1 < q \leq 2$ by using ideas from [8, 7]. In the general case the result is proved by Labutin, by purely analytic methods [12]. Note that $q' = \frac{q}{q-1}$ and $\text{Cap}_{2,q'}$ is the capacity associated to the Sobolev space $W^{2,q'}(\mathbb{R}^N)$.

In [20] we obtain sufficient conditions for the existence of a large solution to

$$-\Delta u + e^u - 1 = 0 \quad \text{in } \Omega, \tag{1.9}$$

expressed in terms of the Hausdorff \mathcal{H}_1^{N-2} -capacity in \mathbb{R}^N , and more precisely

$$\int_0^1 \frac{\mathcal{H}_1^{N-2}(\Omega^c \cap B_r(x))}{r^{N-2}} \frac{dr}{r} = \infty \quad \text{for all } x \in \partial\Omega. \quad (1.10)$$

We refer to [18] for investigation of the initial trace theory of (1.3).

In [9], Evans and Gariepy establish a Wiener criterion for the regularity of a boundary point (in the sense of potential theory) for the heat operator $L = \partial_t - \Delta$ in an arbitrary bounded set of \mathbb{R}^{N+1} . We denote by $\mathfrak{M}(\mathbb{R}^{N+1})$ the set of Radon measures in \mathbb{R}^{N+1} and, for any compact set $K \subset \mathbb{R}^{N+1}$, by $\mathfrak{M}_K(\mathbb{R}^{N+1})$ the subset of $\mathfrak{M}(\mathbb{R}^{N+1})$ of measures with support in K . Their positive cones are respectively denoted by $\mathfrak{M}^+(\mathbb{R}^{N+1})$ and $\mathfrak{M}_K^+(\mathbb{R}^{N+1})$. The capacity used in this criterion is the thermal capacity defined by

$$\text{Cap}_{\mathbb{H}}(K) = \sup\{\mu(K) : \mu \in \mathfrak{M}_K^+(\mathbb{R}^{N+1}), \mathbb{H} * \mu \leq 1\},$$

for any $K \subset \mathbb{R}^{N+1}$ compact, where \mathbb{H} is the heat kernel in \mathbb{R}^{N+1} . It coincides with the parabolic Bessel \mathcal{G}_1 -capacity $\text{Cap}_{\mathcal{G}_1,2}$,

$$\text{Cap}_{\mathcal{G}_1,2}(K) = \sup \left\{ \int_{\mathbb{R}^{N+1}} |f|^2 dx dt : f \in L_+^2(\mathbb{R}^{N+1}), \mathcal{G}_1 * f \geq \chi_K \right\},$$

here \mathcal{G}_1 is the parabolic Bessel kernel of first order, see [21, Remark 4.12]. Garofalo and Lanconelli [10] extend this result to the parabolic operator $L = \partial_t - \text{div}(A(x,t)\nabla)$, where $A(x,t) = (a_{i,j}(x,t))$, $i, j = 1, 2, \dots, N$ is a real, symmetric, matrix-valued function on \mathbb{R}^{N+1} with C^∞ entries satisfying

$$C^{-1}|\xi|^2 \leq \sum_{i,j=1}^N a_{i,j}(x,t)\xi_i\xi_j \leq C|\xi|^2 \quad \forall (x,t) \in \mathbb{R}^{N+1}, \forall \xi \in \mathbb{R}^N,$$

for some constant $C > 0$.

Much less is known concerning the equation

$$\partial_t u - \Delta u + f(u) = 0 \quad (1.11)$$

in a bounded open set $O \subset \mathbb{R}^{N+1}$, where f is a continuous function in \mathbb{R} . Gariepy and Ziemer [11, 24] prove that if there exist $(x_0, t_0) \in \partial_p O$, $l \in \mathbb{R}$ and a weak solution $u \in W^{1,2}(O) \cap L^\infty(O)$ of (1.11) such that $\eta(-l - \varepsilon + u)^+, \eta(l - \varepsilon - u)^+ \in W_0^{1,2}(O)$ for any $\varepsilon > 0$ and $\eta \in C_c^\infty(B_r(x_0) \times (-r^2 + t_0, r^2 + t_0))$ for some $r > 0$, and if there holds

$$\int_0^1 \frac{\text{Cap}_{\mathbb{H}}(O^c \cap (B_\rho(x_0) \times (t_0 - \frac{9}{4}\alpha\rho^2, t_0 - \frac{5}{4}\alpha\rho^2)))}{\rho^N} \frac{d\rho}{\rho} = \infty \text{ for some } \alpha > 0,$$

then $\lim_{(x,t) \rightarrow (x_0, t_0)} u(x,t) = l$. This result is not easy to use because it is not clear whether

(1.11) has a weak solution $u \in W^{1,2}(O)$. In this article we show that (1.11) admits a maximal solution $u \in C^{2,1}(O)$ in an arbitrary bounded open set O , which is constructed by using an approximation of O from inside by dyadic parabolic cubes, provided that f is as in (1.3) and satisfies (1.4).

The main purpose of this article is to extend Labutin's result [12] to the semilinear parabolic equation (1.1). Namely, we give a necessary and a sufficient condition for the existence of solutions to problem (1.1) in a bounded non-cylindrical domain $O \subset \mathbb{R}^{N+1}$, expressed in terms of a Wiener test based upon the parabolic $W_{q'}^{2,1}$ -capacity in \mathbb{R}^{N+1} . We also give a sufficient condition for solving problem (1.2) expressed in terms of a Wiener test based upon the parabolic Hausdorff \mathcal{PH}_ρ^N -capacity. These capacities are defined as follows: if $K \subset \mathbb{R}^{N+1}$ is a compact set, we set

$$\text{Cap}_{2,1,q'}(K) = \inf\{\|\varphi\|_{W_{q'}^{2,1}(\mathbb{R}^{N+1})}^{q'} : \varphi \in S(\mathbb{R}^{N+1}), \varphi \geq 1 \text{ in a neighborhood of } K\},$$

where

$$\|\varphi\|_{W_{q'}^{2,1}(\mathbb{R}^{N+1})} = \|\varphi\|_{L^{q'}(\mathbb{R}^{N+1})} + \left\| \frac{\partial \varphi}{\partial t} \right\|_{L^{q'}(\mathbb{R}^{N+1})} + \|\nabla \varphi\|_{L^{q'}(\mathbb{R}^{N+1})} + \sum_{i,j} \left\| \frac{\partial^2 \varphi}{\partial x_i \partial x_j} \right\|_{L^{q'}(\mathbb{R}^{N+1})},$$

and for a Suslin set $E \subset \mathbb{R}^{N+1}$,

$$\text{Cap}_{2,1,q'}(E) = \sup\{\text{Cap}_{2,1,q'}(D) : D \subset E, D \text{ compact}\}.$$

This capacity has been used in order to obtain estimates expressed with the help of potential that are most helpful for studying quasilinear parabolic equations (see e.g. [3, 4, 21]). Thanks to a result due to Richard and Bagby [2], the capacities $\text{Cap}_{2,1,p}$ and $\text{Cap}_{\mathcal{G}_2,p}$ are equivalent in the sense that, for any Suslin set $K \subset \mathbb{R}^{N+1}$, there holds

$$C^{-1} \text{Cap}_{2,1,q'}(K) \leq \text{Cap}_{\mathcal{G}_2,q'}(K) \leq C \text{Cap}_{2,1,q'}(K),$$

for some $C = C(N, q)$, where $\text{Cap}_{\mathcal{G}_2,q'}$ is the parabolic Bessel \mathcal{G}_2 -capacity, see [21]. For a set $E \subset \mathbb{R}^{N+1}$, we define $\mathcal{PH}_\rho^N(E)$ by

$$\mathcal{PH}_\rho^N(E) = \inf \left\{ \sum_j r_j^N : E \subset \bigcup B_{r_j}(x_j) \times (t_j - r_j^2, t_j + r_j^2), r_j \leq \rho \right\}.$$

It is easy to see that, for $0 < \sigma \leq \rho$ and $E \subset \mathbb{R}^{N+1}$, there holds

$$\mathcal{PH}_\rho^N(E) \leq \mathcal{PH}_\sigma^N(E) \leq C(N) \left(\frac{\rho}{\sigma} \right)^2 \mathcal{PH}_\rho^N(E). \quad (1.12)$$

With these notations, we can state the two main results of this paper.

Theorem 1.1 *Let $N \geq 2$ and $q \geq q_* := \frac{N+2}{N}$. Then*

(i) *The equation*

$$\partial_t u - \Delta u + u^q = 0 \text{ in } O \quad (1.13)$$

admits a large solution if there holds

$$\int_0^1 \frac{\text{Cap}_{2,1,q'}(O^c \cap (B_{\frac{\rho}{30}}(x) \times (t - 30\rho^2, t - \rho^2)))}{\rho^N} \frac{d\rho}{\rho} = \infty, \quad (1.14)$$

for any $(x, t) \in \partial_p O$ and $q > q_$ or $q = q_*$ when $N \geq 3$.*

(ii) *If equation (1.13) admits a large solution, then*

$$\int_0^1 \frac{\text{Cap}_{2,1,q'}(O^c \cap Q_\rho(x, t))}{\rho^N} \frac{d\rho}{\rho} = \infty, \quad (1.15)$$

for any $(x, t) \in \partial_p O$, where $Q_\rho(x, t) = B_\rho(x) \times (t - \rho^2, t)$.

It is an open problem to prove that the maximal solution is unique whenever it exists as it holds in the elliptic case for equation (1.7), see Remark p. 25.

Theorem 1.2 *Let $N \geq 2$. The equation*

$$\partial_t u - \Delta u + e^u - 1 = 0 \text{ in } O \quad (1.16)$$

admits a large solution if there holds

$$\int_0^1 \frac{\mathcal{PH}_1^N(O^c \cap (B_{\frac{\rho}{30}}(x) \times (t - 30\rho^2, t - \rho^2)))}{\rho^N} \frac{d\rho}{\rho} = \infty, \quad (1.17)$$

for any $(x, t) \in \partial_p O$.

From properties of the $W_{q'}^{2,1}$ -capacity, relation (1.14) is satisfied if the following relations hold in which $|\cdot|$ denotes the Lebesgue measure in \mathbb{R}^{N+1} ,

$$\int_0^1 \frac{|O^c \cap (B_{\frac{\rho}{30}}(x) \times (t - 30\rho^2, t - \rho^2))|^{1 - \frac{2q'}{N+2}}}{\rho^N} \frac{d\rho}{\rho} = \infty \quad \text{when } q > q_*,$$

and

$$\int_0^1 \frac{\left(\log_+ \left| O^c \cap (B_{\frac{\rho}{30}}(x) \times (t - 30\rho^2, t - \rho^2)) \right|^{-1} \right)^{-\frac{N}{2}}}{\rho^N} \frac{d\rho}{\rho} = \infty \quad \text{when } q = q_*.$$

Similarly, it follows from properties of the \mathcal{PH}_1^N -capacity that identity (1.17) is verified if

$$\int_0^1 \frac{|O^c \cap (B_{\frac{\rho}{30}}(x) \times (t - 30\rho^2, t - \rho^2))|^{\frac{N}{N+2}}}{\rho^N} \frac{d\rho}{\rho} = \infty.$$

When $O = \{(x, t) \in \mathbb{R}^{N+1} : |x|^2 + \frac{|t|^2}{\lambda} < 1\}$ for some $\lambda > 0$, we see that $\partial O = \partial_p O$. Therefore (1.15) holds for any $(x, t) \in \partial_p O$, and (1.14)-(1.17) hold for any $(x, t) \in \partial_p O \setminus \{(0, \sqrt{\lambda})\}$. However, (1.14) and (1.17) are also valid at $(x, t) = (0, \sqrt{\lambda})$ if $\lambda > 1800^2$, but not valid if $\lambda < 1800^2$.

As a consequence of Theorem 1.1 we derive a sufficient condition for the existence of a large solution of a class of viscous parabolic Hamilton-Jacobi equations.

Theorem 1.3 *Let $q_1 > 1$. If there exists a large solution $v \in C^{2,1}(O)$ of*

$$\partial_t v - \Delta v + v^{q_1} = 0 \quad \text{in } O,$$

then, for any $a, b > 0$, $1 < q < q_1$ and $1 < p < \frac{2q_1}{q_1+1}$, the problem

$$\begin{aligned} \partial_t u - \Delta u + a|\nabla u|^p + bu^q &= 0 & \text{in } O, \\ u &= \infty & \text{on } \partial_p O, \end{aligned} \tag{1.18}$$

admits a solution $u \in C^{2,1}(O)$ which satisfies

$$u(x, t) \geq C \min \left\{ a^{-\frac{1}{p-1}} R^{-\frac{2-p}{p-1} + \frac{2}{\alpha(q_1-1)}}, b^{-\frac{1}{q-1}} R^{-\frac{2}{q-1} + \frac{2}{\alpha(q_1-1)}} \right\} (v(x, t))^{\frac{1}{\alpha}},$$

for all $(x, t) \in O$ where $R > 0$ is such that $O \subset \tilde{Q}_R(x_0, t_0)$, $C = C(N, p, q, q_1) > 0$ and $\alpha = \max \left\{ \frac{2(p-1)}{(q_1-1)(2-p)}, \frac{q-1}{q_1-1} \right\} \in (0, 1)$.

2 Preliminaries

Throughout the paper, we denote

$$Q_\rho(x, t) = B_\rho(x) \times (t - \rho^2, t],$$

and

$$\tilde{Q}_\rho(x, t) = B_\rho(x) \times (t - \rho^2, t + \rho^2),$$

for $(x, t) \in \mathbb{R}^{N+1}$ and $\rho > 0$, and $r_k = 4^{-k}$ for all $k \in \mathbb{Z}$. We also denote $A \lesssim (\gtrsim) B$ if $A \leq (\geq) CB$ for some C depending on some structural constants, $A \asymp B$ if $A \lesssim B \lesssim A$.

Definition 2.1 Let $R \in (0, \infty]$ and $\mu \in \mathfrak{M}^+(\mathbb{R}^{N+1})$. We define R -truncated Riesz parabolic potential \mathbb{I}_2^R of μ by

$$\mathbb{I}_2^R[\mu](x, t) = \int_0^R \frac{\mu(\tilde{Q}_\rho(x, t))}{\rho^N} \frac{d\rho}{\rho} \quad \text{for all } (x, t) \in \mathbb{R}^{N+1},$$

and the R -truncated fractional maximal parabolic potential \mathbb{M}_2^R of μ by

$$\mathbb{M}_2^R[\mu](x, t) = \sup_{0 < \rho < R} \frac{\mu(\tilde{Q}_\rho(x, t))}{\rho^N} \quad \text{for all } (x, t) \in \mathbb{R}^{N+1}.$$

We also set $\mathbb{I}_2^\infty = \mathbb{I}_2$ and $\mathbb{M}_2^\infty = \mathbb{M}_2$. We recall two results in [21].

Theorem 2.2 Let $q > 1, R > 0$ and K be a compact set in \mathbb{R}^{N+1} . There exists $\mu := \mu_K \in \mathfrak{M}^+(\mathbb{R}^{N+1})$ with compact support in K such that

$$\mu(K) \asymp \text{Cap}_{2,1,q'}(K) \asymp \int_{\mathbb{R}^{N+1}} (\mathbb{I}_2^{2R}[\mu])^q dx dt$$

where the constants of equivalence depend on N, q and R . The measure μ_K is called the capacitary measure of K .

Theorem 2.3 For any $R > 0$, there exist positive constants C_1, C_2 such that for any $\mu \in \mathfrak{M}^+(\mathbb{R}^{N+1})$ such that $\|\mathbb{M}_2^R[\mu]\|_{L^\infty(\mathbb{R}^{N+1})} \leq 1$, there holds

$$\frac{1}{|Q|} \int_Q \exp(C_1 \mathbb{I}_2^R[\chi_Q \mu]) dx dt := \int_Q \exp(C_1 \mathbb{I}_2^R[\chi_Q \mu]) dx dt \leq C_2,$$

for all $Q = \tilde{Q}_r(y, s) \subset \mathbb{R}^{N+1}$, $r > 0$, where χ_Q is the indicator function of Q .

Frostman's Lemma in [22, Th. 3.4.27] is at the core of the dual definition of Hausdorff capacities with doubling weight. It is easy to see that it is valid for the parabolic Hausdorff \mathcal{PH}_ρ^N -capacity version. As a consequence we have

Theorem 2.4 There holds

$$\sup \{ \mu(K) : \mu \in \mathfrak{M}^+(\mathbb{R}^{N+1}), \text{supp}(\mu) \subset K, \|\mathbb{M}_2^p[\mu]\|_{L^\infty(\mathbb{R}^{N+1})} \leq 1 \} \asymp \mathcal{PH}_\rho^N(K),$$

for any compact set $K \subset \mathbb{R}^{N+1}$ and $\rho > 0$, where equivalent constant depends on N .

For our purpose, we need the some results about the behavior of the capacity with respect to dilations.

Proposition 2.5 Let $K \subset \overline{\tilde{Q}_{100}(0, 0)}$ be a compact set and $1 < p < \frac{N+2}{2}$. Then

$$\text{Cap}_{2,1,p}(K) \gtrsim |K|^{1-\frac{2p}{N+2}} \quad \text{and} \quad \text{Cap}_{2,1,\frac{N+2}{2}}(K) \gtrsim \left(\log \left(\frac{|\tilde{Q}_{200}(0, 0)|}{|K|} \right) \right)^{-\frac{N}{2}}. \quad (2.1)$$

Furthermore

$$\text{Cap}_{2,1,p}(K_\rho) \asymp \rho^{N+2-2p} \text{Cap}_{2,1,p}(K), \quad (2.2)$$

$$\frac{1}{\text{Cap}_{2,1,\frac{N+2}{2}}(K_\rho)} \asymp \frac{1}{\text{Cap}_{2,1,\frac{N+2}{2}}(K)} + (\log(2/\rho))^{N/2}, \quad (2.3)$$

for any $0 < \rho < 1$, where $K_\rho = \{(\rho x, \rho^2 t) : (x, t) \in K\}$.

Proposition 2.6 *Let $K \subset \overline{\tilde{Q}_1(0,0)}$ be a compact set and $1 < p \leq \frac{N+2}{2}$. Then, there exists a function $\varphi \in C_c^\infty(\tilde{Q}_{3/2}(0,0))$ with $0 \leq \varphi \leq 1$ and $\varphi|_D = 1$ for some open set $D \supset K$ such that*

$$\int_{\mathbb{R}^{N+1}} (|D^2\varphi|^p + |\nabla\varphi|^p + |\varphi|^p + |\partial_t\varphi|^p) dxdt \lesssim \text{Cap}_{2,1,p}(K). \quad (2.4)$$

We will give proofs of the above two propositions in the Appendix.

Let $\{e^{t\Delta}\}_{t \geq 0}$ be the semigroup of contractions in L^p ($1 \leq p < \infty$) generated by Δ . It is wellknown that the solution u of the problem

$$\begin{aligned} \partial_t u - \Delta u &= \mu & \text{in } \tilde{Q}_R(0,0), \\ u &= 0 & \text{on } \partial_p \tilde{Q}_R(0,0), \end{aligned} \quad (2.5)$$

with $\mu \in C^\infty(\tilde{Q}_R(0,0))$, can be expressed by Duhamel's formula

$$u(x,t) = \int_0^t \left(e^{(t-s)\Delta} \mu \right) (x,s) ds \quad \text{for all } (x,t) \in \tilde{Q}_R(0,0).$$

We denote by \mathbb{H} the Gaussian kernel in \mathbb{R}^{N+1} :

$$\mathbb{H}(x,t) = \frac{1}{(4\pi t)^{\frac{N}{2}}} e^{-\frac{|x|^2}{4t}} \chi_{t>0}.$$

We have

$$|u(x,t)| \leq (\mathbb{H} * \mu)(x,t) \quad \text{for all } (x,t) \in \tilde{Q}_R(0,0).$$

In [21, Proof of Proposition 4.8] we show that

$$|(\mathbb{H} * \mu)|(x,t) \leq C_1(N) \mathbb{I}_2^{2R}[|\mu|](x,t) \quad \text{for all } (x,t) \in \tilde{Q}_R(0,0).$$

Here μ is extended by 0 in $(\tilde{Q}_R(0,0))^c$. Thus,

$$\left| \int_0^t \left(e^{(t-s)\Delta} \mu \right) (x,s) ds \right| \leq C_1(N) \mathbb{I}_2^{2R}[|\mu|](x,t) \quad \text{for all } (x,t) \in \tilde{Q}_R(0,0). \quad (2.6)$$

Moreover, we also prove in [21], that if $\mu \geq 0$ then for $(x,t) \in \tilde{Q}_R(0,0)$ and $B_\rho(x) \subset B_R(0)$,

$$\int_0^t \left(e^{(t-s)\Delta} \mu \right) (x,s) ds \geq C_2(N) \sum_{k=0}^{\infty} \frac{\mu(Q_{\frac{\rho_k}{8}}(x, t - \frac{35}{128}\rho_k^2))}{\rho_k^N}, \quad (2.7)$$

with $\rho_k = 4^{-k}\rho$.

It is easy to see that estimates (2.6) and (2.7) also holds for any bounded Radon measure μ in $\tilde{Q}_R(0,0)$. The following result is proved in [3] and [19], and also in [21] in a more general framework.

Theorem 2.7 *Let $q > 1$, $R > 0$ and μ be a bounded Radon measure in $\tilde{Q}_R(0,0)$.*

(i) *If μ is absolutely continuous with respect to $\text{Cap}_{2,1,q'}$ in $\tilde{Q}_R(0,0)$, then there exists a unique weak solution u to equation*

$$\begin{aligned} \partial_t u - \Delta u + |u|^{q-1}u &= \mu & \text{in } \tilde{Q}_R(0,0), \\ u &= 0 & \text{on } \partial_p \tilde{Q}_R(0,0). \end{aligned}$$

(ii) *If $\exp(C_1(N) \mathbb{I}_2^{2R}[|\mu|]) \in L^1(\tilde{Q}_R(0,0))$, then there exists a unique weak solution v to equation*

$$\begin{aligned} \partial_t v - \Delta v + \text{sign}(v)(e^{|v|} - 1) &= \mu & \text{in } \tilde{Q}_R(0,0), \\ v &= 0 & \text{on } \partial_p \tilde{Q}_R(0,0), \end{aligned}$$

where the constant $C_1(N)$ is the one of inequality (2.6).

From estimates (2.6) and (2.7) and using comparison principle we get the estimates from below of the solutions u and v obtained in Theorem 2.7.

Proposition 2.8 *If μ is nonnegative, then the functions u and v of the previous theorem are nonnegative too and satisfy*

$$u(x, t) \geq C_2(N) \sum_{k=0}^{\infty} \frac{\mu(Q_{\frac{\rho_k}{8}}(x, t - \frac{35}{128}\rho_k^2))}{\rho_k^N} - C_1(N)^{q+1} \mathbb{I}_2^{2R} \left[(\mathbb{I}_2^{2R}[\mu])^q \right] (x, t), \quad (2.8)$$

and

$$v(x, t) \geq C_2(N) \sum_{k=0}^{\infty} \frac{\mu(Q_{\frac{\rho_k}{8}}(x, t - \frac{35}{128}\rho_k^2))}{\rho_k^N} - C_1(N) \mathbb{I}_2^{2R} \left[\exp(C_1(N) \mathbb{I}_2^{2R}[\mu]) - 1 \right] (x, t), \quad (2.9)$$

for any $(x, t) \in \tilde{Q}_R(0, 0)$ and $B_\rho(x) \subset B_R(0)$ and $\rho_k = 4^{-k}\rho$.

3 Maximal solutions

In this section we assume that O is an arbitrary non-cylindrical and bounded open set in \mathbb{R}^{N+1} and $q > 1$. We will prove the existence of a maximal solution of

$$\partial_t u - \Delta u + u^q = 0 \quad (3.1)$$

in O . We also get an analogous result when u^q is replaced by $e^u - 1$.

It is easy to see that if u satisfies (3.1) in $\tilde{Q}_r(0, 0)$ ($Q_r(0, 0)$) then $u_a(x, t) = a^{-2/(q-1)}u(ax, a^2t)$ satisfies (3.1) in $\tilde{Q}_{r/a}(0, 0)$ ($Q_{r/a}(0, 0)$) for any $a > 0$. If $X = (x, t) \in O$, the parabolic distance from X to the parabolic boundary $\partial_p O$ of O is defined by

$$d(X, \partial_p O) = \inf_{\substack{(y, s) \in \partial_p O \\ s \leq t}} \max\{|x - y|, (t - s)^{\frac{1}{2}}\}.$$

It is easy to see that there exists $C = C(N, q) > 0$ such that the function V defined by

$$V(x, t) = C \left((\rho^2 + t)^{-\frac{1}{q-1}} + \left(\frac{\rho^2 - |x|^2}{\rho} \right)^{-\frac{2}{q-1}} \right) \quad \text{in } B_\rho(0) \times (-\rho^2, 0),$$

satisfies

$$\partial_t V - \Delta V + V^q \geq 0 \quad \text{in } B_\rho(0) \times (-\rho^2, 0). \quad (3.2)$$

Proposition 3.1 *There exists a maximal solution $u \in C^{2,1}(O)$ of (3.1) and it satisfies*

$$u(x, t) \leq C(d((x, t), \partial_p O))^{-\frac{2}{q-1}} \quad \text{for all } (x, t) \in O, \quad (3.3)$$

for some $C = C(N, q)$.

Proof. Let \mathcal{D}_k , $k \in \mathbb{Z}$ be the collection of all the dyadic parabolic cubes (abridged p -cubes) of the form

$$\{(x_1, \dots, x_N, t) : m_j 2^{-k} \leq x_j \leq (m_j + 1) 2^{-k}, j = 1, \dots, N, m_{N+1} 4^{-k} \leq t \leq (m_{N+1} + 1) 4^{-k}\}$$

where $m_j \in \mathbb{Z}$. The following properties hold,

- a. for each integer k , \mathcal{D}_k is a partition of \mathbb{R}^{N+1} and all p -cubes in \mathcal{D}_k have the same sidelengths.

b. if the interiors of two p -cubes Q in \mathcal{D}_{k_1} and P in \mathcal{D}_{k_2} , denoted $\overset{\circ}{Q}, \overset{\circ}{P}$, have nonempty intersection then either Q is contained in P or Q contains P .

c. Each Q in \mathcal{D}_k is union of 2^{N+2} p -cubes in \mathcal{D}_{k+1} with disjoint interiors.

Let $k_0 \in \mathbb{N}$ be such that $Q \subset O$ for some $Q \in \mathcal{D}_{k_0}$. Set $O_k = \bigcup_{\substack{Q \in \mathcal{D}_k \\ Q \subset O}} Q$, $\forall k \geq k_0$, we

have $O_k \subset O_{k+1}$ and $O = \bigcup_{k \geq k_0} O_k = \bigcup_{k \geq k_0} \overset{\circ}{O}_k$. More precisely, there exist real numbers $a_1, a_2, \dots, a_{n(k)}$ and open sets $\Omega_1, \Omega_2, \dots, \Omega_{n(k)}$ in \mathbb{R}^N such that

$$a_i < a_i + 4^{-k} \leq a_{i+1} < a_{i+1} + 4^{-k} \quad \text{for } i = 1, \dots, n(k) - 1,$$

and

$$\overset{\circ}{O}_k = \bigcup_{i=1}^{n(k)-1} (\Omega_i \times (a_i, a_i + 4^{-k}]) \bigcup (\Omega_{n(k)} \times (a_{n(k)}, a_{n(k)} + 4^{-k})).$$

For $k \geq k_0$, we claim that there exists a solution $u_k \in C^{2,1}(\overset{\circ}{O}_k)$ to problem

$$\begin{aligned} \partial_t u_k - \Delta u_k + u_k^q &= 0 & \text{in } \overset{\circ}{O}_k, \\ u_k(x, t) &\rightarrow \infty & \text{as } d((x, t), \partial_p \overset{\circ}{O}_k) \rightarrow 0. \end{aligned} \quad (3.4)$$

Indeed, by [6, 15] for $m > 0$, one can find nonnegative solutions $v_i \in C^{2,1}(\Omega_i \times (a_i, a_i + 4^{-k}]) \cap C(\overline{\Omega}_i \times [a_i, a_i + 4^{-k}])$ for $i = 1, \dots, n(k)$ to equations

$$\begin{aligned} \partial_t v_1 - \Delta v_1 + v_1^q &= 0 & \text{in } \Omega_1 \times (a_1, a_1 + 4^{-k}), \\ v_1(x, t) &= m & \text{on } \partial \Omega_1 \times (a_1, a_1 + 4^{-k}), \\ v_1(x, a_1) &= m & \text{in } \Omega_1, \end{aligned}$$

and

$$\begin{aligned} \partial_t v_i - \Delta v_i + v_i^q &= 0 & \text{in } \Omega_i \times (a_i, a_i + 4^{-k}), \\ v_i(x, t) &= m & \text{on } \partial \Omega_i \times (a_i, a_i + 4^{-k}), \\ v_i(x, a_i) &= m_i & \text{in } \Omega_i, \end{aligned}$$

where

$$m_i = \begin{cases} m & \text{in } \Omega_i \\ m \chi_{\Omega_i \setminus \Omega_{i-1}}(x) + v_{i-1}(x, a_{i-1} + 4^{-k}) \chi_{\Omega_{i-1}}(x) & \text{otherwise.} \end{cases} \quad \text{if } a_i > a_{i-1} + 4^{-k},$$

Clearly,

$$u_{k,m} = v_i \quad \text{in } \Omega_i \times (a_i, a_i + 4^{-k}] \quad \text{for } i = 1, \dots, n(k)$$

is a solution in $C^{2,1}(\overset{\circ}{O}_k) \cap C(O_k)$ to equation

$$\begin{aligned} \partial_t u_{k,m} - \Delta u_{k,m} + u_{k,m}^q &= 0 & \text{in } \overset{\circ}{O}_k, \\ u_{k,m} &= m & \text{on } \partial_p \overset{\circ}{O}_k. \end{aligned}$$

Moreover, for $(x, t) \in \overset{\circ}{O}_k$, we see that $B_{\frac{d}{2}}(x) \times (t - \frac{d^2}{4}, t) \subset \overset{\circ}{O}_k$ where $d = d((x, t), \partial_p \overset{\circ}{O}_k)$. From (3.2), we verify that

$$U(y, s) := V(y - x, s - t) = C \left((\rho^2 + s - t)^{-\frac{1}{q-1}} + \left(\frac{\rho^2 - |x - y|^2}{\rho} \right)^{-\frac{2}{q-1}} \right)$$

with $\rho = d/2$, satisfies

$$\partial_t U - \Delta U + U^q \geq 0 \quad \text{in } B_{\frac{d}{2}}(x) \times (t - \frac{d^2}{4}, t). \quad (3.5)$$

Applying the comparison principle we get

$$u_{k,m}(y, s) \leq U(y, s) \quad \text{in } B_{\frac{d}{2}}(x) \times (t - \frac{d^2}{4}, t],$$

which implies

$$u_{k,m}(x, t) \leq C \left(d((x, t), \partial_p \mathring{O}_k) \right)^{-\frac{2}{q-1}} \quad \text{for all } (x, t) \in \mathring{O}_k. \quad (3.6)$$

From this, we obtain also uniform local bounds for $\{u_{k,m}\}_m$. By standard regularity theory see, [6, 15], $\{u_{k,m}\}_m$ is uniformly locally bounded in $C^{2,1}$. Hence, up to a subsequence, $u_{k,m} \rightarrow u_k$ $C_{\text{loc}}^{1,0}(\mathring{O}_k)$ as $m \rightarrow \infty$. We derive that u_k is a weak solution of (3.4) in \mathring{O}_k , which satisfies $u_k(x, t) \rightarrow \infty$ as $d((x, t), \partial_p \mathring{O}_k) \rightarrow 0$ and

$$u_k(x, t) \leq C \left(d((x, t), \partial_p \mathring{O}_k) \right)^{-\frac{2}{q-1}} \quad \text{for all } (x, t) \in \mathring{O}_k.$$

Let $m > 0$ and $k \geq k_0$. Since $u_{k+1,m} \leq m$ in O_k and $O_k \subset O_{k+1}$, it follows by the comparison principle applied to $u_{k+1,m}$ and $u_{k,m}$ in the following $n(k)$ sub-domains of \mathring{O}_k : $\Omega_1 \times (a_1, a_1 + 4^{-k})$, $\Omega_2 \times (a_2, a_2 + 4^{-k})$, ..., $\Omega_{n(k)} \times (a_{n(k)}, a_{n(k)} + 4^{-k})$, that $u_{k+1,m} \leq u_{k,m}$ in \mathring{O}_k , and thus $u_{k+1} \leq u_k$ in \mathring{O}_k by letting $m \rightarrow \infty$. In particular, $\{u_k\}_k$ is uniformly locally bounded in L_{loc}^∞ . We use the same compactness property as above to infer that $u_k \rightarrow u$ as $k \rightarrow \infty$. Then u is a solution of (3.1) and it satisfies (3.3). By construction u is the maximal solution. \blacksquare

Remark 3.2 Let $R \geq 2r \geq 2$, K be a compact subset in $\overline{\tilde{Q}_r(0, 0)}$. As in the proof of Proposition 3.1, we can show that there exists a maximal solution of

$$\begin{aligned} \partial_t u - \Delta u + u^q &= 0 & \text{in } \tilde{Q}_R(0, 0) \setminus K, \\ u &= 0 & \text{on } \partial_p \tilde{Q}_R(0, 0), \end{aligned} \quad (3.7)$$

which satisfies

$$u(x, t) \leq C(d((x, t), \partial_p(\tilde{Q}_R(0, 0) \setminus K)))^{-\frac{2}{q-1}} \quad \forall (x, t) \in \tilde{Q}_R(0, 0) \setminus K, \quad (3.8)$$

for some $C = C(N, q)$. Furthermore, assume K_1, K_2, \dots, K_m are compact subsets in $\overline{\tilde{Q}_r(0, 0)}$ and $K = K_1 \cup \dots \cup K_m$. Let u, u_1, \dots, u_m be the maximal solutions of (3.7) in $\tilde{Q}_R(0, 0) \setminus K, \tilde{Q}_R(0, 0) \setminus K_1, \tilde{Q}_R(0, 0) \setminus K_2, \dots, \tilde{Q}_R(0, 0) \setminus K_m$, respectively, then

$$u \leq \sum_{j=1}^m u_j \quad \text{in } \tilde{Q}_R(0, 0) \setminus K. \quad (3.9)$$

Remark 3.3 If the equation (3.1) admits a large solution for some $q > 1$, then for any $1 < q_1 < q$, the equation

$$\partial_t u - \Delta u + u^{q_1} = 0 \quad \text{in } O \quad (3.10)$$

admits also a large solution.

Indeed, assume that u is a large solution of (3.1) and v the maximal solution of (3.10). Take $R > 0$ such that $O \subset B_R(0) \times (-R^2, R^2)$, then the function V defined by

$$V(x, t) = (q - 1)^{-\frac{1}{q-1}} (2R^2 + t)^{-\frac{1}{q-1}},$$

satisfies (3.1). It follows for all $(x, t) \in O$

$$u(x, t) \geq \inf_{(y, s) \in O} V(x, t) \geq (q-1)^{-\frac{1}{q-1}} R^{-\frac{2}{q-1}} =: a_0.$$

Then $\tilde{u} = a_0^{\frac{q-q_1}{q_1-1}} u$ is a subsolution of (3.10). Therefore $v \geq a_0^{\frac{q-q_1}{q_1-1}} u$ in O , thus v is a large solution.

Remark 3.4 (Sub-critical case) Assume that $1 < q < q_*$. It is easy to check that the function

$$U(x, t) = \frac{C}{t^{\frac{1}{q-1}}} e^{-\frac{|x|^2}{4t}} \chi_{t>0} \quad (3.11)$$

is a subsolution of (3.1) in $\mathbb{R}^{N+1} \setminus \{(0, 0)\}$, where $C = \left(\frac{2}{q-1} - \frac{N}{2}\right)^{\frac{1}{q-1}}$.

Therefore, the maximal solution u of (3.1) in O verifies

$$u(x, t) \geq C \frac{1}{(t-s)^{\frac{1}{q-1}}} e^{-\frac{|x-y|^2}{4(t-s)}} \chi_{t>s}, \quad (3.12)$$

for all $(x, t) \in O$ and $(y, s) \in O^c$.

If for any $(x, t) \in \partial_p O$ there exist $\varepsilon \in (0, 1)$ and a decreasing sequence $\{\delta_n\} \subset (0, 1)$ converging to 0 as $n \rightarrow \infty$ such that $(B_{\delta_n}(x) \times (-\delta_n^2 + t, -\varepsilon\delta_n^2 + t)) \cap O^c \neq \emptyset$ for any $n \in \mathbb{N}$, then u is a large solution. For proving this, we need to show that

$$\lim_{\rho \rightarrow 0} \inf_{O \cap (B_\rho(x) \times (-\rho^2 + t, \rho^2 + t))} u = \infty.$$

Let $0 < \rho < \sqrt{\frac{\varepsilon}{2}}\delta_1$ and $n \in \mathbb{N}$ such that $\sqrt{\frac{\varepsilon}{2}}\delta_{n+1} \leq \rho < \sqrt{\frac{\varepsilon}{2}}\delta_n$.

Since $(B_{\delta_n}(x) \times (-\delta_n^2 + t, -\varepsilon\delta_n^2 + t)) \cap O^c \neq \emptyset$, there is $(x_n, t_n) \in O^c$ such that $|x_n - x| < \delta_n$ and $-\delta_n^2 + t < t_n < -\varepsilon\delta_n^2 + t$. So if $(y, s) \in O \cap (B_\rho(x) \times (-\rho^2 + t, \rho^2 + t))$ then $|y - x_n| < (\sqrt{\varepsilon} + 1)\delta_n$ and $\frac{\varepsilon}{2}\delta_n^2 < s - t_n < (\varepsilon + 1)\delta_n^2$. Hence, thanks to (3.12) we have for any $(y, s) \in O \cap (B_\rho(x) \times (-\rho^2 + t, \rho^2 + t))$

$$u(y, s) \geq C \frac{1}{(s - t_n)^{\frac{1}{q-1}}} e^{-\frac{|y-x_n|^2}{4(s-t_n)}} \geq C(\varepsilon + 1)^{-\frac{1}{q-1}} e^{-\frac{(\sqrt{\varepsilon}+1)^2}{2\varepsilon}} \delta_n^{-\frac{2}{q-1}},$$

which implies

$$\inf_{O \cap (B_\rho(x) \times (-\rho^2 + t, \rho^2 + t))} u \geq C(\varepsilon + 1)^{-\frac{1}{q-1}} e^{-\frac{(\sqrt{\varepsilon}+1)^2}{2\varepsilon}} \delta_n^{-\frac{2}{q-1}} \rightarrow \infty \quad \text{as } \rho \rightarrow 0.$$

Remark 3.5 Note that if $u \in C^{2,1}(O)$ is a solution of (3.1) for some $q > 1$ then, for $a, b > 0$ and $1 < p \leq 2$, the function $v = b^{-\frac{1}{q-1}} u$ is a super-solution of

$$\partial_t v - \Delta v + a|\nabla v|^p + bv^q = 0 \quad \text{in } O. \quad (3.13)$$

Thus, we can apply the argument of the previous proof, with equation (3.1) replaced by (3.13), and deduce that there exists a maximal solution $v \in C^{2,1}(O)$ of (3.13) satisfying

$$v(x, t) \leq C b^{-\frac{1}{q-1}} (d((x, t), \partial_p O))^{-\frac{2}{q-1}} \quad \text{for all } (x, t) \in O.$$

Furthermore, if $1 < q < q_*$, $q = \frac{2p}{p+1}$, $a, b > 0$ then the function U expressed by (3.11) in Remark 3.4 is a subsolution of (3.13) in $\mathbb{R}^{N+1} \setminus \{(0, 0)\}$, provided the explicit constant C

given therein is replaced by some $C = C(N, p, q, a, b)$. Therefore, we conclude that every maximal solution of $v \in C^{2,1}(O)$ of (3.13) satisfy

$$v(x, t) \geq C \frac{1}{(t-s)^{\frac{1}{q-1}}} e^{-\frac{|x-y|^2}{4(t-s)}} \chi_{t>s} \quad (3.14)$$

for all $(x, t) \in O$ and $(y, s) \in \partial_p O$.

Arguing as in Remark 3.4, if for any $(x, t) \in \partial_p O$ there exist $\varepsilon \in (0, 1)$ and a decreasing sequence $\{\delta_n\} \subset (0, 1)$ converging to 0 as $n \rightarrow \infty$ such that $(B_{\delta_n}(x) \times (-\delta_n^2 + t, -\varepsilon\delta_n^2 + t)) \cap O^c \neq \emptyset$ for any $n \in \mathbb{N}$, then v is a large solution.

Next, we consider the following equation

$$\partial_t u - \Delta u + e^u - 1 = 0. \quad (3.15)$$

It is easy to see that the two functions

$$V_1(t) = -\log\left(\frac{t+\rho^2}{1+\rho^2}\right) \quad \text{and} \quad V_2(x) = C - 2\log\left(\frac{\rho^2 - |x|^2}{\rho}\right)$$

satisfy

$$V_1' + e^{V_1} - 1 \geq 0 \quad \text{in } (-\rho^2, 0],$$

and

$$-\Delta V_2 + e^{V_2} - 1 \geq 0 \quad \text{in } B_\rho(0),$$

for some $C = C(N)$. Using $e^a + e^b \leq e^{a+b} - 1$ for $a, b \geq 0$, we obtain that $V_1 + V_2$ is a supersolution of equation (3.15) in $B_\rho(0) \times (-\rho^2, 0]$. By the same argument as in Proposition 3.1 and the estimate of the above supersolution, we infer the following:

Proposition 3.6 *There exists a maximal solution $u \in C^{2,1}(O)$ of*

$$\partial_t u - \Delta u + e^u - 1 = 0 \quad \text{in } O, \quad (3.16)$$

and it satisfies

$$u(x, t) \leq C - \log\left(\frac{(d((x, t), \partial_p O))^3}{4 + (d((x, t), \partial_p O))^2}\right) \quad \text{for all } (x, t) \in O, \quad (3.17)$$

for some $C = C(N)$.

The next three propositions will be useful to prove Theorem 1.1-(ii).

Proposition 3.7 *Let $K \subset \overline{\tilde{Q}_1(0, 0)}$ be a compact set and $q > 1$, $R \geq 100$. Let u be a solution of (3.7) in $\tilde{Q}_R(0, 0) \setminus K$ and φ as in Proposition 2.6 with $p = q'$. Set $\xi = (1 - \varphi)^{2q'}$. Then,*

$$\int_{\tilde{Q}_R(0, 0)} u(|\Delta \xi| + |\nabla \xi| + |\partial_t \xi|) dx dt \lesssim \text{Cap}_{2,1,q'}(K), \quad (3.18)$$

$$u(x, t) \lesssim \text{Cap}_{2,1,q'}(K) + R^{-\frac{2}{q-1}} \quad \text{for any } (x, t) \in \tilde{Q}_{R/5}(0, 0) \setminus \tilde{Q}_2(0, 0), \quad (3.19)$$

and

$$\int_{\tilde{Q}_2(0, 0)} u \xi dx dt \lesssim \text{Cap}_{2,1,q'}(K) + R^{-\frac{2}{q-1}}, \quad (3.20)$$

where the constants in above inequalities depend only on N and q .

Proof. *Step 1.* We claim that

$$\int_{\tilde{Q}_R(0,0)} u^q \xi dxdt \lesssim \text{Cap}_{2,1,q'}(K). \quad (3.21)$$

Actually, using integration by parts and the Green formula, one has

$$\begin{aligned} \int_{\tilde{Q}_R(0,0)} u^q \xi dxdt &= - \int_{\tilde{Q}_R(0,0)} \partial_t u \xi dxdt + \int_{\tilde{Q}_R(0,0)} \xi \Delta u dxdt \\ &= \int_{\tilde{Q}_R(0,0)} u \partial_t \xi dxdt + \int_{\tilde{Q}_R(0,0)} u \Delta \xi dxdt + \int_{-R^2}^{R^2} \int_{\partial B_R(0)} \left(\xi \frac{\partial u}{\partial \nu} - u \frac{\partial \xi}{\partial \nu} \right) dSdt, \end{aligned}$$

where ν is the outer normal unit vector on $\partial B_R(0)$. Clearly,

$$\frac{\partial u}{\partial \nu} \leq 0 \quad \text{and} \quad \frac{\partial \xi}{\partial \nu} = 0 \quad \text{on} \quad \partial B_R(0).$$

Thus,

$$\begin{aligned} \int_{\tilde{Q}_R(0,0)} u^q \xi dxdt &\leq \int_{\tilde{Q}_R(0,0)} u |\partial_t \xi| dxdt + \int_{\tilde{Q}_R(0,0)} u |\Delta \xi| dxdt \\ &\leq 2q' \int_{\tilde{Q}_R(0,0)} u (1-\varphi)^{2q'-1} |\partial_t \varphi| dxdt + 2q'(2q'-1) \int_{\tilde{Q}_R(0,0)} u (1-\varphi)^{2q'-2} |\nabla \varphi|^2 dxdt \\ &\quad + 2q' \int_{\tilde{Q}_R(0,0)} u (1-\varphi)^{2q'-1} |\Delta \varphi| dxdt \\ &\leq 2q' \int_{\tilde{Q}_R(0,0)} u \xi^{1/q} |\partial_t \varphi| dxdt + 2q'(2q'-1) \int_{\tilde{Q}_R(0,0)} u \xi^{1/q} |\nabla \varphi|^2 dxdt \\ &\quad + 2q' \int_{\tilde{Q}_R(0,0)} u \xi^{1/q} |\Delta \varphi| dxdt. \end{aligned} \quad (3.22)$$

In the last inequality, we have used the fact that $(1-\phi)^{2q'-1} \leq (1-\phi)^{2q'-2} = \xi^{1/q}$. Hence, by Hölder's inequality,

$$\begin{aligned} \int_{\tilde{Q}_R(0,0)} u^q \xi dxdt &\lesssim \int_{\tilde{Q}_R(0,0)} |\partial_t \varphi|^{q'} dxdt + \int_{\tilde{Q}_R(0,0)} |\nabla \varphi|^{2q'} dxdt \\ &\quad + \int_{\tilde{Q}_R(0,0)} |\Delta \varphi|^{q'} dxdt. \end{aligned}$$

By the Gagliardo-Nirenberg inequality,

$$\begin{aligned} \int_{\tilde{Q}_R(0,0)} |\nabla \varphi|^{2q'} dxdt &\lesssim \|\varphi\|_{L^\infty(\tilde{Q}_R(0,0))}^{q'} \int_{\tilde{Q}_R(0,0)} |D^2 \varphi|^{q'} dxdt \\ &\lesssim \int_{\tilde{Q}_R(0,0)} |D^2 \varphi|^{q'} dxdt. \end{aligned}$$

Hence, we find

$$\int_{\tilde{Q}_R(0,0)} u^q \xi dxdt \lesssim \int_{\tilde{Q}_R(0,0)} (|\partial_t \varphi|^{q'} + |D^2 \varphi|^{q'}) dxdt,$$

and derive (3.21) from (2.4). In view of (3.22), we also obtain

$$\int_{\tilde{Q}_R(0,0)} u (|\Delta \xi| + |\partial_t \xi|) dxdt \lesssim \text{Cap}_{2,1,q'}(K),$$

and

$$\int_{\tilde{Q}_R(0,0)} u |\nabla \xi| dx dt \lesssim \text{Cap}_{2,1,q'}(K),$$

since

$$\begin{aligned} \int_{\tilde{Q}_R(0,0)} u |\nabla \xi| dx dt &= 2q' \int_{\tilde{Q}_R(0,0)} u \xi^{(2q'-1)/2q'} |\nabla \varphi| dx dt \\ &\leq 2q' \int_{\tilde{Q}_R(0,0)} u \xi^{1/q} |\nabla \varphi| dx dt \\ &\lesssim \int_{\tilde{Q}_R(0,0)} u^q \xi dx dt + \int_{\tilde{Q}_R(0,0)} |\nabla \varphi|^{q'} dx dt. \end{aligned}$$

It yields (3.18).

Step 2. Relation (3.19) holds. Let η be a cut off function on $\tilde{Q}_{R/4}(0,0)$ with respect to $\tilde{Q}_{R/3}(0,0)$ such that $|\partial_t \eta| + |D^2 \eta| \lesssim R^{-2}$ and $|\nabla \eta| \lesssim R^{-1}$. We have

$$\partial_t(\eta \xi u) - \Delta(\eta \xi u) = F \in C_c(\tilde{Q}_{R/3}(0,0)).$$

Hence, we can write

$$(\eta \xi u)(x, t) = \int_{\mathbb{R}^N} \int_{-\infty}^t \frac{1}{(4\pi(t-s))^{\frac{N}{2}}} e^{-\frac{|x-y|^2}{4(t-s)}} F(y, s) ds dy \quad \forall (x, t) \in \mathbb{R}^{N+1}.$$

Now, we fix $(x, t) \in \tilde{Q}_{R/5}(0,0) \setminus \tilde{Q}_2(0,0)$. Since $\text{supp}\{|\nabla \eta|\} \cap \text{supp}\{|\nabla \xi|\} = \emptyset$ and

$$\begin{aligned} F &= \eta \xi (\partial_t u - \Delta u) - 2(\eta \nabla \xi + \xi \nabla \eta) \nabla u + (\xi \partial_t \eta + \eta \partial_t \xi - 2\nabla \eta \nabla \xi - \Delta \eta \xi - \eta \Delta \xi) u \\ &\leq -2(\eta \nabla \xi + \xi \nabla \eta) \nabla u + (\xi \partial_t \eta + \eta \partial_t \xi - \xi \Delta \eta - \eta \Delta \xi) u, \end{aligned}$$

there holds

$$\begin{aligned} u(x, t) &= (\eta \xi u)(x, t) \leq -2 \int_{\mathbb{R}^N} \int_{-\infty}^t \frac{1}{(4\pi(t-s))^{\frac{N}{2}}} e^{-\frac{|x-y|^2}{4(t-s)}} (\eta \nabla \xi + \xi \nabla \eta) \nabla u ds dy \\ &\quad + \int_{\mathbb{R}^N} \int_{-\infty}^t \frac{1}{(4\pi(t-s))^{\frac{N}{2}}} e^{-\frac{|x-y|^2}{4(t-s)}} (\eta \partial_t \xi - \eta \Delta \xi) u ds dy \\ &\quad + \int_{\mathbb{R}^N} \int_{-\infty}^t \frac{1}{(4\pi(t-s))^{\frac{N}{2}}} e^{-\frac{|x-y|^2}{4(t-s)}} (\partial_t \eta \xi - \xi \Delta \eta) u ds dy. \\ &= I_1 + I_2 + I_3. \end{aligned}$$

By integration by parts,

$$\begin{aligned} I_1 &= 2(4\pi)^{-N/2} \int_{-\infty}^t \int_{\mathbb{R}^N} \frac{(x-y)}{2(t-s)^{(N+2)/2}} e^{-\frac{|x-y|^2}{4(t-s)}} (\eta \nabla \xi + \xi \nabla \eta) u dy ds \\ &\quad + 2(4\pi)^{-N/2} \int_{-\infty}^t \int_{\mathbb{R}^N} \frac{1}{(t-s)^{N/2}} e^{-\frac{|x-y|^2}{4(t-s)}} (\xi \Delta \eta + \eta \Delta \xi) u dy ds. \end{aligned}$$

Note that

$$\begin{aligned} \frac{1}{(t-s)^{N/2}} e^{-\frac{|x-y|^2}{4(t-s)}} &\lesssim \left(\max\{|x-y|, |t-s|^{1/2}\} \right)^{-N}, \\ \left| \frac{(x-y)}{2(t-s)^{(N+2)/2}} e^{-\frac{|x-y|^2}{4(t-s)}} \right| &\lesssim \left(\max\{|x-y|, |t-s|^{1/2}\} \right)^{-N-1}, \end{aligned}$$

and

$$\begin{aligned} \max\{|x-y|, |t-s|^{1/2}\} &\gtrsim 1 \quad \forall (y, s) \in \text{supp}\{|D^\alpha \xi|\} \cup \text{supp}\{|\partial_t \xi|\}, \\ \max\{|x-y|, |t-s|^{1/2}\} &\gtrsim R \quad \forall (y, s) \in \text{supp}\{|D^\alpha \eta|\} \cup \text{supp}\{|\partial_t \eta|\} \quad \forall |\alpha| \geq 1. \end{aligned}$$

We deduce

$$\begin{aligned} I_1 &\lesssim \int_{\mathbb{R}^{N+1}} \left(\max\{|x-y|, |t-s|^{1/2}\} \right)^{-N-1} (\eta |\nabla \xi| + \xi |\nabla \eta|) u \, dy ds \\ &\quad + \int_{\mathbb{R}^{N+1}} \left(\max\{|x-y|, |t-s|^{1/2}\} \right)^{-N} (\xi |\Delta \eta| + \eta |\Delta \xi|) u \, dy ds \\ &\lesssim \int_{\mathbb{R}^{N+1}} (|\nabla \xi| + |\Delta \xi|) u \, dy ds + \int_{\tilde{Q}_{R/3}(0,0) \setminus \tilde{Q}_{R/4}(0,0)} (R^{-N-1} |\nabla \eta| + R^{-N} |\Delta \eta|) u \, dy ds \\ &\lesssim \int_{\mathbb{R}^{N+1}} (|\nabla \xi| + |\Delta \xi|) u \, dy ds + \sup_{\tilde{Q}_{R/3}(0,0) \setminus \tilde{Q}_{R/4}(0,0)} u, \\ I_2 &\lesssim \int_{\mathbb{R}^{N+1}} \left(\max\{|x-y|, |t-s|^{1/2}\} \right)^{-N} (|\partial_t \xi| + |\Delta \xi|) u \, dy ds \\ &\lesssim \int_{\mathbb{R}^{N+1}} (|\partial_t \xi| + |\Delta \xi|) u \, dy ds, \end{aligned}$$

and

$$\begin{aligned} I_3 &\lesssim \int_{\mathbb{R}^{N+1}} \left(\max\{|x-y|, |t-s|^{1/2}\} \right)^{-N} (|\partial_t \eta| + |\Delta \eta|) u \, dy ds \\ &\lesssim \int_{\tilde{Q}_{R/3}(0,0) \setminus \tilde{Q}_{R/4}(0,0)} R^{-N} (|\partial_t \eta| + |\Delta \eta|) u \, dy ds \\ &\lesssim \sup_{\tilde{Q}_{R/3}(0,0) \setminus \tilde{Q}_{R/4}(0,0)} u. \end{aligned}$$

Hence,

$$u(x, t) \leq I_1 + I_2 + I_3 \lesssim \int_{\mathbb{R}^{N+1}} (|\partial_t \xi| + |\nabla \xi| + |\Delta \xi|) u \, dy ds + \sup_{\tilde{Q}_{R/3}(0,0) \setminus \tilde{Q}_{R/4}(0,0)} u.$$

Combining this inequality with (3.18) and (3.8), we obtain (3.19).

Step 3. End of the proof. Let θ be a cut off function on $\tilde{Q}_3(0,0)$ with respect to $\tilde{Q}_4(0,0)$. As above, we have for any $(x, t) \in \mathbb{R}^{N+1}$

$$\begin{aligned} (\theta \xi u)(x, t) &\lesssim \int_{\mathbb{R}^{N+1}} \left(\max\{|x-y|, |t-s|^{1/2}\} \right)^{-N-1} (\theta |\nabla \xi| + \xi |\nabla \theta|) u \, dy ds \\ &\quad + \int_{\mathbb{R}^{N+1}} \left(\max\{|x-y|, |t-s|^{1/2}\} \right)^{-N} (\theta |\Delta \xi| + \xi |\Delta \theta|) u \, dy ds \\ &\quad + \int_{\mathbb{R}^{N+1}} \left(\max\{|x-y|, |t-s|^{1/2}\} \right)^{-N} (\theta |\partial_t \xi| + \theta |\Delta \xi|) u \, dy ds \\ &\quad + \int_{\mathbb{R}^{N+1}} \left(\max\{|x-y|, |t-s|^{1/2}\} \right)^{-N} (\xi |\partial_t \theta| + \xi |\Delta \theta|) u \, dy ds. \end{aligned}$$

Hence, by Fubini theorem,

$$\begin{aligned} \int_{\tilde{Q}_2(0,0)} \eta u \, dx dt &= \int_{\tilde{Q}_2(0,0)} \theta \eta u \, dx dt \\ &\lesssim A \int_{\mathbb{R}^{N+1}} (\theta |\nabla \xi| + \xi |\nabla \theta| + \theta |\Delta \xi| + \xi |\Delta \theta| + \theta |\partial_t \xi| + \xi |\partial_t \theta|) u \, dy ds \\ &\lesssim \int_{\mathbb{R}^{N+1}} (|\partial_t \xi| + |\nabla \xi| + |\Delta \xi|) u \, dy ds + \sup_{\tilde{Q}_4(0,0) \setminus \tilde{Q}_3(0,0)} u, \end{aligned}$$

where

$$A = \sup_{(y,s) \in \tilde{Q}_4(0,0)} \int_{\tilde{Q}_2(0,0)} ((\max\{|x-y|, |t-s|^{1/2}\})^{-N} + (\max\{|x-y|, |t-s|^{1/2}\})^{-N-1}) dx dt.$$

Therefore we obtain (3.20) from (3.18) and (3.19). \blacksquare

Proposition 3.8 *Let $0 < \varepsilon < 1$, $K \subset \{(x, t) : \varepsilon < \max\{|x|, |t|^{1/2}\} < 1\}$ be a compact set and u the maximal solution of (3.7) in $\tilde{Q}_R(0,0) \setminus K$ with $R \geq 100$. Then*

$$\sup_{\tilde{Q}_{\varepsilon/4}(0,0)} u \lesssim \sum_{j=-2}^{j_\varepsilon-2} \frac{\text{Cap}_{2,1,q'}(K \cap \tilde{Q}_{\rho_j}(0,0))}{\rho_j^N} + j_\varepsilon R^{-\frac{2}{q-1}} \quad \text{if } q > q_*, \quad (3.23)$$

and

$$\sup_{\tilde{Q}_{\varepsilon/4}(0,0)} u \lesssim \sum_{j=0}^{j_\varepsilon} \frac{\text{Cap}_{2,1,q'}(K_j)}{\rho_j^N} + j_\varepsilon R^{-\frac{2}{q-1}} \quad \text{if } q = q_*, \quad (3.24)$$

where $\rho_j = 2^{-j}$, $K_j = \{(x/\rho_{j+3}, t/\rho_{j+3}^2) : (x, t) \in K \cap \tilde{Q}_{\rho_{j-2}}(0,0)\}$ and $j_\varepsilon \in \mathbb{N}$ is such that $\rho_{j_\varepsilon} \leq \varepsilon < \rho_{j_\varepsilon-1}$.

Proof. For $j \in \mathbb{N}$, we define $S_j = \{x : \rho_j \leq \max\{|x|, |t|^{1/2}\} \leq \rho_{j-1}\}$. Fix any $1 \leq j \leq j_\varepsilon$. We cover S_j by $L = L(N) \in \mathbb{N}^*$ closed cylinders

$$\overline{\tilde{Q}_{\rho_{j+3}}(x_{k,j}, t_{k,j})}, \quad k = 1, \dots, L(N),$$

where $(x_{k,j}, t_{k,j}) \in S_j$.

For $k = 1, \dots, L(N)$, let $u_j, u_{k,j}$ be the maximal solutions of (3.7) where K is replaced by $K \cap S_j$ and $K \cap \tilde{Q}_{\rho_{j+3}}(x_{k,j}, t_{k,j})$, respectively. Clearly the function $\tilde{u}_{k,j}$ defined by

$$\tilde{u}_{k,j}(x, t) = \rho_{j+3}^{\frac{2}{q-1}} u_{k,j}(\rho_{j+3}x + x_{k,j}, \rho_{j+3}^2t + t_{k,j})$$

is the maximal solution of (3.7) provided $(K_{k,j}, \tilde{Q}_{R/\rho_{j+3}}(-x_{k,j}/\rho_{j+3}, -t_{k,j}/\rho_{j+3}^2))$ with

$$K_{k,j} = \{(y/\rho_{j+3}, s/\rho_{j+3}^2) : (y, s) \in -(x_{k,j}, t_{k,j}) + K \cap \overline{\tilde{Q}_{\rho_{j+3}}(x_{k,j}, t_{k,j})}\} \subset \overline{\tilde{Q}_1(0,0)}$$

is replacing $(K, \tilde{Q}_R(0,0))$. Let $\bar{u}_{k,j}$ be the maximal solution of (3.7) with $(K, \tilde{Q}_R(0,0))$ replaced by $(K_{k,j}, \tilde{Q}_{2R/\rho_{j+3}}(0,0))$. Since $\tilde{Q}_{R/\rho_{j+3}}(-x_{k,j}/\rho_{j+3}, -t_{k,j}/\rho_{j+3}^2) \subset \tilde{Q}_{2R/\rho_{j+3}}(0,0)$, then, by the comparison principle as in the proof of Proposition 3.1, we get $\tilde{u}_{k,j} \leq \bar{u}_{k,j}$ in $\tilde{Q}_{R/\rho_{j+3}}(-x_{k,j}/\rho_{j+3}, -t_{k,j}/\rho_{j+3}^2) \setminus K_{k,j}$, and thus

$$\tilde{u}_{k,j}(x, t) \lesssim \text{Cap}_{2,1,q'}(K_{k,j}) + (R/\rho_{j+3})^{-\frac{2}{q-1}},$$

for any $(x, t) \in (\tilde{Q}_{2R/(5\rho_{j+3})}(0,0) \cap \tilde{Q}_{R/\rho_{j+3}}(-x_{k,j}/\rho_{j+3}, -t_{k,j}/\rho_{j+3}^2)) \setminus \tilde{Q}_2(0,0) = D$.

Fix $(x_0, t_0) \in \tilde{Q}_{\varepsilon/4}(0,0)$. Clearly, $((x_0 - x_{k,j})/\rho_{j+3}, (t_0 - t_{k,j})/\rho_{j+3}^2) \in D$, hence

$$\begin{aligned} u_{k,j}(x_0, t_0) &= \rho_{j+3}^{-\frac{2}{q-1}} \tilde{u}_{k,j}((x_0 - x_{k,j})/\rho_{j+3}, (t_0 - t_{k,j})/\rho_{j+3}^2) \\ &\lesssim \frac{\text{Cap}_{2,1,q'}(K_{k,j})}{\rho_j^{\frac{2}{q-1}}} + R^{-\frac{2}{q-1}}. \end{aligned}$$

Therefore, using (3.9) in Remark 3.2 and the fact that

$$\text{Cap}_{2,1,q'}(K_{k,j}) = \text{Cap}_{2,1,q'}(K_{k,j} + (x_{k,j}/\rho_{j+3}, t_{k,j}/\rho_{j+3}^2)) \leq \text{Cap}_{2,1,q'}(K_j),$$

we derive

$$\begin{aligned} u(x_0, t_0) &\leq \sum_{j=1}^{j_\varepsilon} u_j(x_0, t_0) \leq \sum_{j=1}^{j_\varepsilon} \sum_{k=1}^{L(N)} u_{k,j}(x_0, t_0) \\ &\lesssim \sum_{j=0}^{j_\varepsilon} \frac{\text{Cap}_{2,1,q'}(K_j)}{\rho_j^{\frac{2}{q-1}}} + j_\varepsilon R^{-\frac{2}{q-1}}, \end{aligned}$$

which yields (3.24). If $q > q_*$, then by (2.2) in Proposition 2.5, we have

$$\text{Cap}_{2,1,q'}(K_j) \lesssim \rho_{j+3}^{-N-2+2q'} \text{Cap}_{2,1,q'}(K \cap \tilde{Q}_{\rho_{j-2}}(0,0)),$$

which implies (3.23). ■

Proposition 3.9 *Let K, u, ξ be as in Proposition 3.7. For any compact set K_0 in $\overline{\tilde{Q}_1(0,0)}$ with positive measure $|K_0|$, there exists $\varepsilon = \varepsilon(N, q, |K_0|) > 0$ such that*

$$\text{Cap}_{2,1,q'}(K) \leq \varepsilon \Rightarrow \inf_{K_0} u \lesssim \int_{\tilde{Q}_2(0,0)} u \xi dx dt,$$

where the constant in the inequality \lesssim depends on K_0 . In particular,

$$\text{Cap}_{2,1,q'}(K) \leq \varepsilon \Rightarrow \inf_{K_0} u \lesssim \text{Cap}_{2,1,q'}(K) + R^{-\frac{2}{q-1}}. \quad (3.25)$$

Proof. It is enough to prove that there exists $\varepsilon > 0$ such that

$$\text{Cap}_{2,1,q'}(K) \leq \varepsilon \Rightarrow |K_1| \geq 1/2 |K_0|, \quad (3.26)$$

where $K_1 = \{(x, t) \in K_0 : \xi(x, t) \geq 1/2\}$. By (2.1) in Proposition 2.5, we have the following estimates

$$|K_0 \setminus K_1|^{1-\frac{2q'}{N+2}} \lesssim \text{Cap}_{2,1,q'}(K_0 \setminus K_1),$$

if $q > q_*$, and

$$\left(\log \left(\frac{|\tilde{Q}_{200}(0,0)|}{|K_0 \setminus K_1|} \right) \right)^{-\frac{N}{2}} \lesssim \text{Cap}_{2,1,q'}(K_0 \setminus K_1),$$

if $q = q_*$. On the other hand,

$$\begin{aligned} \text{Cap}_{2,1,q'}(K_0 \setminus K_1) &= \text{Cap}_{2,1,q'}(\{K_0 : \varphi > 1 - (1/2)^{1/(2q')}\}) \\ &\leq (1 - (1/2)^{1/(2q')})^{-q'} \int_{\mathbb{R}^{N+1}} \left(|D^2 \varphi|^{q'} + |\nabla \varphi|^{q'} + |\varphi|^{q'} + |\partial_t \varphi|^{q'} \right) dx dt \\ &\lesssim \text{Cap}_{2,1,q'}(K), \end{aligned}$$

where φ is in Proposition 3.7. Henceforth, one can find $\varepsilon = \varepsilon(N, q, |K_0|) > 0$ such that

$$\text{Cap}_{2,1,q'}(K) \leq \varepsilon \Rightarrow |K_0 \setminus K_1| \leq 1/2 |K_0|.$$

This implies (3.26). ■

4 Large solutions

In the first part of this section, we prove theorem 1.1-(ii), then we prove theorems 1.1-(i) and 1.2. At end we apply our result to a parabolic viscous Hamilton-Jacobi equation.

4.1 Proof of Theorem 1.1-(ii)

Let $R_0 \geq 4$ such that $O \subset \subset \tilde{Q}_{R_0}(0, 0)$. Assume that the equation (1.13) has a large solution u . We claim that (1.15) holds with $(x, t) \in \partial_p O$, and without loss of generality, we can assume $(x, t) = (0, 0)$. Set $K = \tilde{Q}_{2R_0}(0, 0) \setminus O$ and define

$$\begin{aligned} T_j &= \{x : \rho_{j+1} \leq \max\{|x|, |t|^{1/2}\} \leq \rho_j, t \leq 0\}, \\ \tilde{T}_j &= \{x : \rho_{j+3} \leq \max\{|x|, |t|^{1/2}\} \leq \rho_{j-2}, t \leq 0\}. \end{aligned}$$

Here $\rho_j = 2^{-j}$. For $j \geq 3$, let u_1, u_2, u_3, u_4 be the maximal solutions of (3.7) when K is replaced by $K \cap \overline{Q_{\rho_{j+3}}(0, 0)}$, $K \cap \tilde{T}_j$, $(K \cap \overline{Q_1(0, 0)}) \setminus Q_{\rho_{j-2}}(0, 0)$ and $K \setminus Q_1(0, 0)$ respectively and $R \geq 100R_0$. From (3.9) in Remark 3.2, we can assert that

$$u \leq u_1 + u_2 + u_3 + u_4 \quad \text{in } O \cap \{(x, t) \in \mathbb{R}^{N+1} : t \leq 0\}.$$

Thus,

$$\inf_{T_j} u \leq \|u_1\|_{L^\infty(T_j)} + \|u_3\|_{L^\infty(T_j)} + \|u_4\|_{L^\infty(T_j)} + \inf_{T_j} u_2. \quad (4.1)$$

Case 1: $q > q_*$. By (3.8) in Remark 3.2,

$$\|u_4\|_{L^\infty(T_j)} \lesssim 1. \quad (4.2)$$

By (3.23) in Proposition 3.8,

$$\|u_3\|_{L^\infty(T_j)} \lesssim \sum_{i=-2}^{j-4} \frac{\text{Cap}_{2,1,q'}(K \cap Q_{\rho_i}(0, 0))}{\rho_i^N} + jR^{-\frac{2}{q-1}}. \quad (4.3)$$

Since $(x, t) \mapsto \bar{u}_1(x, t) = \rho_{j+3}^{2/(q-1)} u_1(\rho_{j+3}x, \rho_{j+3}^2 t)$ is the maximal solution of (3.7) when $(K, \tilde{Q}_R(0, 0))$ is replaced by $(\{(y/\rho_{j+3}, s/\rho_{j+3}^2) : (y, s) \in K \cap \overline{Q_{\rho_{j+3}}(0, 0)}\}, \tilde{Q}_{R/\rho_{j+3}}(0, 0))$, we derive

$$\|\bar{u}_1\|_{L^\infty(T_{-3})} \lesssim \frac{\text{Cap}_{2,1,q'}(K \cap \overline{Q_{\rho_{j+2}}(0, 0)})}{\rho_j^{N+2-2q'}} + (R/\rho_{j+3})^{-\frac{2}{q-1}},$$

thanks to (3.19) in Proposition 3.7 and (2.2) in Proposition 2.5, from which follows

$$\|u_1\|_{L^\infty(T_j)} \lesssim \frac{\text{Cap}_{2,1,q'}(K \cap \overline{Q_{\rho_{j+2}}(0, 0)})}{\rho_j^N} + R^{-\frac{2}{q-1}}. \quad (4.4)$$

Since $(x, t) \mapsto \bar{u}_2(x, t) = \rho_{j-2}^{2/(q-1)} u_2(\rho_{j-2}x, \rho_{j-2}^2 t)$ is the maximal solution of (3.7) when the couple $(K, \tilde{Q}_R(0, 0))$ is replaced by $(\{(y/\rho_{j-2}, s/\rho_{j-2}^2) : (y, s) \in K \cap \tilde{T}_j\}, \tilde{Q}_{R/\rho_{j-2}}(0, 0))$, Proposition 3.9 and relation (2.2) in Proposition 2.5 yield

$$\frac{\text{Cap}_{2,1,q'}(K \cap \tilde{T}_j)}{\rho_{j-2}^{N+2-2q'}} \leq \varepsilon \Rightarrow \inf_{T_2} \bar{u}_2 \lesssim \frac{\text{Cap}_{2,1,q'}(K \cap \tilde{T}_j)}{\rho_{j-2}^{N+2-2q'}} + (R/\rho_{j-2})^{-\frac{2}{q-1}},$$

which implies

$$\frac{\text{Cap}_{2,1,q'}(K \cap \overline{Q_{\rho_{j-3}}(0, 0)})}{\rho_{j-2}^{N+2-2q'}} \leq \varepsilon \Rightarrow \inf_{T_j} u_2 \lesssim \frac{\text{Cap}_{2,1,q'}(K \cap \overline{Q_{\rho_{j-3}}(0, 0)})}{\rho_{j-2}^N} + R^{-\frac{2}{q-1}}, \quad (4.5)$$

for some $\varepsilon = \varepsilon(N, q) > 0$.

First, we assume that there exists $J \in \mathbb{N}$, $J \geq 10$ such that

$$\frac{\text{Cap}_{2,1,q'}(K \cap \overline{Q_{\rho_{j-3}}(0, 0)})}{\rho_{j-2}^{N+2-2q'}} \leq \varepsilon \quad \forall j \geq J.$$

Then, from (4.1) and (4.2), (4.3), (4.4), (4.5), we have

$$\inf_{T_j} u \lesssim \sum_{i=-2}^{j+2} \frac{\text{Cap}_{2,1,q'}(K \cap Q_{\rho_i}(0,0))}{\rho_i^N} + jR^{-\frac{2}{q-1}} + 1,$$

for any $j \geq J$, and letting $R \rightarrow \infty$,

$$\inf_{T_j} u \lesssim \sum_{i=-2}^{j+2} \frac{\text{Cap}_{2,1,q'}(K \cap Q_{\rho_i}(0,0))}{\rho_i^N} + 1.$$

Since $\inf_{T_j} u \rightarrow \infty$ as $j \rightarrow \infty$, we get

$$\sum_{i=0}^{\infty} \frac{\text{Cap}_{2,1,q'}(K \cap Q_{\rho_i}(0,0))}{\rho_i^N} = \infty,$$

which implies that (1.15) holds with $(x, t) = (0, 0)$.

Alternatively, assume that for infinitely many j

$$\frac{\text{Cap}_{2,1,q'}(K \cap Q_{\rho_{j-3}}(0,0))}{\rho_{j-2}^{N+2-2q'}} > \varepsilon,$$

then,

$$\frac{\text{Cap}_{2,1,q'}(K \cap Q_{\rho_{j-3}}(0,0))}{\rho_{j-2}^N} > \rho_{j-2}^{2-2q'} \varepsilon \rightarrow \infty \quad \text{when } j \rightarrow \infty.$$

We also derive that (1.15) holds with $(x, t) = (0, 0)$. This proves the case $q > q_*$.

Case 2: $q = q_*$. Similarly to Case 1, we have: for $j \geq 6$

$$\|u_4\|_{L^\infty(T_j)} \lesssim 1, \tag{4.6}$$

$$\|u_3\|_{L^\infty(T_j)} \lesssim \sum_{i=0}^{j-2} \frac{\text{Cap}_{2,1,q'}(K_j)}{\rho_i^N} + jR^{-\frac{2}{q-1}}, \tag{4.7}$$

$$\|u_1\|_{L^\infty(T_j)} \lesssim \frac{\text{Cap}_{2,1,q'}(K_j)}{\rho_j^N} + R^{-\frac{2}{q-1}}, \tag{4.8}$$

$$\text{Cap}_{2,1,q'}(K_{j-5}) \leq \varepsilon \Rightarrow \inf_{T_j} u_2 \lesssim \frac{\text{Cap}_{2,1,q'}(K_{j-5})}{\rho_j^N} + R^{-\frac{2}{q-1}}, \tag{4.9}$$

where $K_j = \{(x/\rho_{j+3}, t/\rho_{j+3}^2) : (x, t) \in K \cap Q_{\rho_{j-3}}(0,0)\}$ and $\varepsilon = \varepsilon(N) > 0$. From (2.2) in Proposition 2.5, we have

$$\frac{1}{\text{Cap}_{2,1,q'}(K \cap Q_{\rho_{j-3}}(0,0))} \leq \frac{c}{\text{Cap}_{2,1,q'}(K_j)} + cj^{N/2}$$

for any $j \geq 4$ where $c = c(N)$. If there are infinitely many $j \geq 4$ such that

$$\text{Cap}_{2,1,q'}(K \cap Q_{\rho_{j-3}}(0,0)) > \frac{1}{2cj^{N/2}},$$

then (1.15) holds with $(x, t) = (0, 0)$ since

$$\frac{\text{Cap}_{2,1,q'}(K \cap Q_{\rho_{j-3}}(0,0))}{\rho_{j-3}^N} > \frac{2^{j-3}}{2cj^{N/2}} \rightarrow \infty \quad \text{when } j \rightarrow \infty.$$

Now, we assume that there exists $J \geq 6$ such that

$$\text{Cap}_{2,1,q'}(K \cap Q_{\rho_{j-3}}(0,0)) \leq \frac{1}{2cj^{N/2}} \quad \forall j \geq J.$$

Then,

$$\text{Cap}_{2,1,q'}(K_j) \leq 2c\text{Cap}_{2,1,q'}(K \cap Q_{\rho_{j-3}}(0,0)) \quad \forall j \geq J.$$

This leads to

$$\text{Cap}_{2,1,q'}(K_j) \leq 2c\text{Cap}_{2,1,q'}(K \cap Q_{\rho_{j-3}}(0,0)) \leq \varepsilon \quad \forall j \geq J' + J,$$

for some $J' = J'(N)$. Hence, from (4.6)-(4.9) we have, for any $j \geq J' + J + 3$,

$$\begin{aligned} \|u_4\|_{L^\infty(T_j)} &\lesssim 1, \\ \|u_3\|_{L^\infty(T_j)} &\lesssim \sum_{i=J'+J+1}^{j-2} \frac{\text{Cap}_{2,1,q'}(K \cap Q_{\rho_{i-3}}(0,0))}{\rho_i^N} + C(J' + J) + jR^{-\frac{2}{q-1}}, \\ \|u_1\|_{L^\infty(T_j)} &\lesssim \frac{\text{Cap}_{2,1,q'}(K \cap Q_{\rho_{j-3}}(0,0))}{\rho_j^N} + R^{-\frac{2}{q-1}}, \\ \inf_{T_j} u_2 &\lesssim \frac{\text{Cap}_{2,1,q'}(K \cap Q_{\rho_{j-8}}(0,0))}{\rho_j^N} + R^{-\frac{2}{q-1}}, \end{aligned}$$

where $C(J' + J) = \sum_{i=0}^{J'+J} \frac{\text{Cap}_{2,1,q'}(K_j)}{\rho_i^N}$.

Consequently we derive

$$\inf_{T_j} u \lesssim \sum_{i=0}^j \frac{\text{Cap}_{2,1,q'}(K \cap Q_{\rho_i}(0,0))}{\rho_i^N} + C(J' + J) + 1 + jR^{-\frac{2}{q-1}} \quad \forall j \geq J' + J + 3$$

from (4.1). Letting $R \rightarrow \infty$ and $j \rightarrow \infty$ we obtain

$$\sum_{i=0}^{\infty} \frac{\text{Cap}_{2,1,q'}(K \cap Q_{\rho_i}(0,0))}{\rho_i^N} = \infty,$$

i.e. (1.15) holds with $(x, t) = (0, 0)$. This completes the proof of Theorem 1.1-(ii).

4.2 Proof of Theorem 1.1-(i) and Theorem 1.2

Fix $(x_0, t_0) \in \partial_p O$. We can assume that $(x_0, t_0) = 0$. Let $\delta \in (0, 1/100)$. For $(y_0, s_0) \in (B_\delta(0) \times (-\delta^2, \delta^2)) \cap O$, we set

$$M_k = O^c \cap \left(\overline{B_{r_{k+2}}(y_0)} \times [s_0 - (73 + \frac{1}{2})r_{k+2}^2, s_0 - (70 + \frac{1}{2})r_{k+2}^2] \right),$$

and

$$S_k = \{(x, t) : r_{k+1} \leq \max\{|x - y_0|, |t - s_0|^{\frac{1}{2}}\} < r_k\} \quad \text{for } k = 1, 2, \dots,$$

where $r_k = 4^{-k}$. Note that $M_k = \emptyset$ for k large enough and $M_k \subset S_k$ for all k . Let $R_0 \geq 4$ such that $O \subset \subset \tilde{Q}_{R_0}(0, 0)$. By Theorems 2.2 and 2.4 and estimate (1.12) there exist two sequences $\{\mu_k\}_k$ and $\{\nu_k\}_k$ of nonnegative Radon measures such that

$$\text{supp}(\mu_k) \subset M_k, \quad \text{supp}(\nu_k) \subset M_k, \quad (4.10)$$

$$\mu_k(M_k) \asymp \text{Cap}_{2,1,q'}(M_k) \asymp \int_{\mathbb{R}^{N+1}} \left(\mathbb{I}_2^{2R_0}[\mu_k] \right)^q dx dt \quad (4.11)$$

and

$$\nu_k(M_k) \asymp \mathcal{PH}_1^N(M_k), \quad \|\mathbb{M}_1^{2R_0}[\nu_k]\|_{L^\infty(\mathbb{R}^{N+1})} \leq 1 \quad \text{for } k = 1, 2, \dots, \quad (4.12)$$

where the constants of equivalence depend on N, q, R_0 .

Take $\varepsilon > 0$ such that $\exp\left(C_1 \mathbb{I}_2^{2R_0}[\sum_{k=1}^\infty \nu_k]\right) \in L^1(\tilde{Q}_{R_0}(0, 0))$, in which expression the constant $C_1 = C_1(N)$ is the one of inequality (2.6). By Theorem 2.7 and Proposition 2.8, there exist two nonnegative solutions U_1, U_2 of problems

$$\begin{aligned} \partial_t U_1 - \Delta U_1 + U_1^q &= \varepsilon \sum_{k=1}^\infty \mu_k && \text{in } \tilde{Q}_{R_0}(0, 0), \\ U_1 &= 0 && \text{on } \partial_p \tilde{Q}_{R_0}(0, 0), \end{aligned}$$

and

$$\begin{aligned} \partial_t U_2 - \Delta U_2 + e^{U_2} - 1 &= \varepsilon \sum_{k=1}^\infty \nu_k && \text{in } \tilde{Q}_{R_0}(0, 0), \\ U_2 &= 0 && \text{on } \partial_p \tilde{Q}_{R_0}(0, 0), \end{aligned}$$

respectively which satisfy

$$\begin{aligned} U_1(y_0, z_0) &\gtrsim \sum_{i=0}^\infty \sum_{k=1}^\infty \varepsilon \frac{\mu_k(B_{\frac{r_i}{8}}(y_0) \times (s_0 - \frac{37}{128}r_i^2, s_0 - \frac{35}{128}r_i^2))}{r_i^N} \\ &\quad - \mathbb{I}_2^{2R_0} \left[\left(\mathbb{I}_2^{2R_0} \left[\varepsilon \sum_{k=1}^\infty \mu_k \right] \right)^q \right] (y_0, s_0) =: A, \end{aligned} \quad (4.13)$$

and

$$\begin{aligned} U_2(y_0, z_0) &\gtrsim \sum_{i=0}^\infty \sum_{k=1}^\infty \varepsilon \frac{\nu_k(B_{\frac{r_i}{8}}(y_0) \times (s_0 - \frac{37}{128}r_i^2, s_0 - \frac{35}{128}r_i^2))}{r_i^N} \\ &\quad - \mathbb{I}_2^{2R_0} \left[\exp \left(C_1 \mathbb{I}_2^{2R_0} \left[\varepsilon \sum_{k=1}^\infty \nu_k \right] \right) - 1 \right] (y_0, s_0) =: B, \end{aligned} \quad (4.14)$$

and $U_1, U_2 \in C^{2,1}(O)$.

Let u_1, u_2 be the maximal solutions of equations (3.1) and (3.16) respectively.

We have $u_1(y_0, s_0) \geq U_1(y_0, s_0)$ and $u_2(y_0, s_0) \geq U_2(y_0, s_0)$. Now, we claim that

$$A \gtrsim \sum_{k=1}^\infty \frac{\text{Cap}_{2,1,q'}(M_k)}{r_k^N}, \quad (4.15)$$

and

$$B \gtrsim -c_1(R_0) + \sum_{k=1}^\infty \frac{\mathcal{PH}_1^N(M_k)}{r_k^N}. \quad (4.16)$$

Proof of assertion (4.15). From (4.11) we have

$$A \gtrsim \varepsilon \sum_{k=1}^\infty \frac{\text{Cap}_{2,1,q'}(M_k)}{r_k^N} - \varepsilon^q A_0, \quad (4.17)$$

with

$$A_0 = \mathbb{I}_2^{2R_0} \left[\left(\mathbb{I}_2^{2R_0} \left[\sum_{k=1}^\infty \mu_k \right] \right)^q \right] (y_0, s_0).$$

Take $i_0 \in \mathbb{Z}$ such that $r_{i_0+1} < \max\{2R_0, 1\} \leq r_{i_0}$. Then

$$\begin{aligned}
A_0 &\lesssim \sum_{i=i_0}^{\infty} r_i^{-N} \int_{\tilde{Q}_{r_i}(y_0, s_0)} \left(\mathbb{I}_2^{2R_0} \left[\sum_{k=1}^{\infty} \mu_k \right] \right)^q dx dt \\
&= \sum_{i=i_0}^{\infty} \sum_{j=i}^{\infty} r_i^{-N} \int_{S_j} \left(\mathbb{I}_2^{2R_0} \left[\sum_{k=1}^{\infty} \mu_k \right] \right)^q dx dt \\
&= \sum_{j=k_0}^{\infty} \sum_{i=i_0}^j r_i^{-N} \int_{S_j} \left(\mathbb{I}_2^{2R_0} \left[\sum_{k=1}^{\infty} \mu_k \right] \right)^q dx dt \\
&\lesssim \sum_{j=i_0}^{\infty} r_j^{-N} \int_{S_j} \left(\mathbb{I}_2^{2R_0} \left[\sum_{k=1}^{\infty} \mu_k \right] \right)^q dx dt.
\end{aligned}$$

Here we have used the fact that $\sum_{i=i_0}^j r_i^{-N} \leq \frac{4}{3} r_j^{-N}$ for all j .

If we set $\mu_k \equiv 0$ for all $i_0 - 1 \leq k \leq 0$, the previous inequality becomes

$$\begin{aligned}
A_0 &\lesssim \sum_{j=i_0}^{\infty} r_j^{-N} \int_{S_j} \left(\mathbb{I}_2^{2R_0} \left[\mu_j + \sum_{k=i_0-1}^{j-1} \mu_k + \sum_{k=j+1}^{\infty} \mu_k \right] \right)^q dx dt \\
&\lesssim \sum_{j=i_0}^{\infty} r_j^{-N} \int_{S_j} \left(\mathbb{I}_2^{2R_0} [\mu_j] \right)^q dx dt \\
&\quad + \sum_{j=i_0}^{\infty} r_j^2 \left(\sum_{k=i_0-1}^{j-1} \|\mathbb{I}_2^{2R_0} [\mu_k]\|_{L^\infty(S_j)} \right)^q \\
&\quad + \sum_{j=i_0}^{\infty} r_j^2 \left(\sum_{k=j+1}^{\infty} \|\mathbb{I}_2^{2R_0} [\mu_k]\|_{L^\infty(S_j)} \right)^q \\
&= A_1 + A_2 + A_3.
\end{aligned} \tag{4.18}$$

Using (4.11) we obtain

$$A_1 \leq \sum_{k=1}^{\infty} \frac{\text{Cap}_{2,1,q'}(M_k)}{r_k^N}. \tag{4.19}$$

Next, using (4.10) we have for any $(x, t) \in S_j$

$$\mathbb{I}_2^{2R_0} [\mu_k](x, t) = \int_{r_{j+1}}^{2R_0} \frac{\mu_k(\tilde{Q}_\rho(x, t))}{\rho^N} \frac{d\rho}{\rho} \lesssim \frac{\mu_k(\mathbb{R}^{N+1})}{r_j^N} \tag{4.20}$$

if $k \geq j+1$, and

$$\mathbb{I}_2^{2R_0} [\mu_k](x, t) = \int_{r_{k+1}}^{2R_0} \frac{\mu_k(\tilde{Q}_\rho(x, t))}{\rho^N} \frac{d\rho}{\rho} \lesssim \frac{\mu_k(\mathbb{R}^{N+1})}{r_k^N} \tag{4.21}$$

if $k \leq j-1$. Thus,

$$A_2 \lesssim \sum_{j=i_0}^{\infty} r_j^2 \left(\sum_{k=i_0-1}^{j-1} \frac{\mu_k(\mathbb{R}^{N+1})}{r_k^N} \right)^q,$$

and

$$A_3 \lesssim \sum_{j=i_0}^{\infty} r_j^{2-Nq} \left(\sum_{k=j+1}^{\infty} \mu_k(\mathbb{R}^{N+1}) \right)^q.$$

Noticing that $(a+b)^q - a^q \leq q(a+b)^{q-1}b$ for any $a, b \geq 0$, we get

$$\begin{aligned}
(1-4^{-2}) \sum_{j=i_0}^{\infty} r_j^2 \left(\sum_{k=i_0-1}^{j-1} \frac{\mu_k(\mathbb{R}^{N+1})}{r_k^N} \right)^q \\
= \sum_{j=i_0}^{\infty} r_j^2 \left(\sum_{k=i_0-1}^{j-1} \frac{\mu_k(\mathbb{R}^{N+1})}{r_k^N} \right)^q - \sum_{j=i_0+1}^{\infty} r_j^2 \left(\sum_{k=i_0-1}^{j-2} \frac{\mu_k(\mathbb{R}^{N+1})}{r_k^N} \right)^q \\
\leq \sum_{j=i_0}^{\infty} q r_j^2 \left(\sum_{k=i_0-1}^{j-1} \frac{\mu_k(\mathbb{R}^{N+1})}{r_k^N} \right)^{q-1} \frac{\mu_{j-1}(\mathbb{R}^{N+1})}{r_{j-1}^N}.
\end{aligned}$$

Similarly, we also have

$$\begin{aligned}
(1-4^{2-Nq}) \sum_{j=i_0}^{\infty} r_j^{2-Nq} \left(\sum_{k=j+1}^{\infty} \mu_k(\mathbb{R}^{N+1}) \right)^q \\
\leq \sum_{j=i_0}^{\infty} q r_j^{2-Nq} \left(\sum_{k=j+1}^{\infty} \mu_k(\mathbb{R}^{N+1}) \right)^{q-1} \mu_{j+1}(\mathbb{R}^{N+1}).
\end{aligned}$$

Therefore,

$$\begin{aligned}
A_2 + A_3 &\lesssim \sum_{j=i_0}^{\infty} r_j^2 \left(\sum_{k=i_0-1}^{j-1} \frac{\mu_k(\mathbb{R}^{N+1})}{r_k^N} \right)^{q-1} \frac{\mu_{j-1}(\mathbb{R}^{N+1})}{r_{j-1}^N} \\
&\quad + \sum_{j=i_0}^{\infty} r_j^{2-Nq} \left(\sum_{k=j+1}^{\infty} \mu_k(\mathbb{R}^{N+1}) \right)^{q-1} \mu_{j+1}(\mathbb{R}^{N+1}).
\end{aligned}$$

Since $\mu_k(\mathbb{R}^{N+1}) \lesssim r_k^{N+2-2q'}$ if $q > q_*$ and $\mu_k(\mathbb{R}^{N+1}) \lesssim \min\{k^{-\frac{1}{q-1}}, 1\}$ if $q = q_*$ for any k , we infer that

$$r_j^2 \left(\sum_{k=i_0-1}^{j-1} \frac{\mu_k(\mathbb{R}^{N+1})}{r_k^N} \right)^{q-1} \lesssim 1,$$

and

$$r_j^{2-Nq} \left(\sum_{k=j+1}^{\infty} \mu_k(\mathbb{R}^{N+1}) \right)^{q-1} \lesssim r_{j+1}^{-N} \quad \text{for any } j.$$

In the case $q = q_*$ we assume $N \geq 3$ in order to ensure that

$$\sum_{j=1}^{\infty} \mu_j(\mathbb{R}^{N+1}) \lesssim \sum_{k=1}^{\infty} k^{-\frac{1}{q-1}} < \infty.$$

This leads to

$$A_2 + A_3 \lesssim \sum_{k=1}^{\infty} \frac{\mu_k(\mathbb{R}^{N+1})}{r_k^N}.$$

Combining this with (4.19) and (4.18), we deduce

$$A_0 \lesssim \sum_{k=1}^{\infty} \frac{\text{Cap}_{2,1,q'}(M_k)}{r_k^N}.$$

Consequently, we obtain (4.15) from (4.17), for ε small enough.

Proof of assertion (4.16). From (4.12) we get

$$B \gtrsim \varepsilon \sum_{k=1}^{\infty} \frac{\mathcal{PH}_1^N(M_k)}{r_k^N} - B_0,$$

where

$$B_0 = \mathbb{I}_2^{2R_0} \left[\exp \left(C_1 \mathbb{I}_2^{2R_0} [\varepsilon \sum_{k=1}^{\infty} \nu_k] \right) - 1 \right] (y_0, s_0).$$

We show that

$$B_0 \leq c(N, q, R_0) \quad \text{for } \varepsilon \text{ small enough.} \quad (4.22)$$

In fact, as above we have

$$B_0 \lesssim \sum_{j=i_0}^{\infty} r_j^{-N} \int_{S_j} \exp \left(C_1 \varepsilon \mathbb{I}_2^{2R_0} [\sum_{k=1}^{\infty} \nu_k] \right) dx dt.$$

Consequently,

$$\begin{aligned} B_0 &\lesssim \sum_{j=i_0}^{\infty} r_j^{-N} \int_{S_j} \exp \left(3C_1 \varepsilon \mathbb{I}_2^{2R_0} [\nu_j] \right) dx dt \\ &\quad + \sum_{j=i_0}^{\infty} r_j^2 \exp \left(3C_1 \varepsilon \sum_{k=i_0-1}^{j-1} \|\mathbb{I}_2^{2R_0} [\nu_k]\|_{L^\infty(S_j)} \right) \\ &\quad + \sum_{j=i_0}^{\infty} r_j^2 \exp \left(3C_1 \varepsilon \sum_{k=j+1}^{\infty} \|\mathbb{I}_2^{2R_0} [\nu_k]\|_{L^\infty(S_j)} \right) \\ &= B_1 + B_2 + B_3. \end{aligned} \quad (4.23)$$

Here we have used the convexity inequality $3 \exp(a + b + c) \leq \exp(3a) + \exp(3b) + \exp(3c)$ for all real numbers a, b, c .

By Theorem 2.3, we have

$$\int_{S_j} \exp \left(3C_1 \varepsilon \mathbb{I}_2^{2R_0} [\nu_j] \right) dx dt \lesssim r_j^{N+2} \quad \text{for all } j,$$

for $\varepsilon > 0$ small enough. Hence,

$$B_1 \lesssim \sum_{j=i_0}^{\infty} r_j^2 \lesssim (\max\{2R_0, 1\})^2. \quad (4.24)$$

Note that estimates (4.20) and (4.21) are also true with ν_k ; we deduce

$$\begin{aligned} B_2 + B_3 &\lesssim \sum_{j=i_0}^{\infty} r_j^2 \exp \left(c_2 \varepsilon \sum_{k=i_0-1}^{j-1} \frac{\mu_k(\mathbb{R}^{N+1})}{r_k^N} \right) \\ &\quad + \sum_{j=i_0}^{\infty} r_j^2 \exp \left(c_2 \varepsilon \sum_{k=j+1}^{\infty} \frac{\mu_k(\mathbb{R}^{N+1})}{r_j^N} \right). \end{aligned}$$

From (4.12) we have $\mu_k(\mathbb{R}^{N+1}) \lesssim r_k^N$ for all k , therefore

$$\begin{aligned} B_2 + B_3 &\lesssim \sum_{j=i_0}^{\infty} r_j^2 \exp(c_3 \varepsilon(j - i_0)) + \sum_{j=i_0}^{\infty} r_j^2 \exp(c_3 \varepsilon) \\ &\lesssim \sum_{j=i_0}^{\infty} \exp(c_3 \varepsilon(j - i_0) - 4 \log(2)j) + r_{i_0}^2 \\ &\leq c_4(N, q, R_0) \quad \text{for } \varepsilon \text{ small enough.} \end{aligned}$$

Combining this with (4.24) and (4.23) we obtain (4.22).

This implies straightforwardly $\exp\left(C_1 \varepsilon \mathbb{I}_2^{2R_0}[\sum_{k=1}^{\infty} \nu_k]\right) \in L^1(\tilde{Q}_{R_0}(0, 0))$.

We conclude that for any $(y_0, s_0) \in (B_\delta(0) \times (-\delta^2, \delta^2)) \cap O$,

$$u_1(y_0, s_0) \gtrsim \sum_{k=1}^{\infty} \frac{\text{Cap}_{2,1,q'}(M_k(y_0, s_0))}{r_k^N},$$

and

$$u_2(y_0, s_0) \gtrsim -c_1(R_0) + \sum_{k=1}^{\infty} \frac{\mathcal{PH}_1^N(M_k(y_0, s_0))}{r_k^N},$$

where $r_k = 4^{-k}$ and

$$M_k(y_0, s_0) = O^c \cap \left(\overline{B_{r_{k+2}}(y_0)} \times [s_0 - (73 + \frac{1}{2})r_{k+2}^2, s_0 - (70 + \frac{1}{2})r_{k+2}^2] \right).$$

If we take $r_{k_\delta+4} \leq \delta < r_{k_\delta+3}$, we have for $1 \leq k \leq k_\delta$

$$\begin{aligned} M_k(y_0, s_0) &\supset O^c \cap \left(B_{r_{k+2}-\delta}(0) \times \left(\delta^2 - (73 + \frac{1}{2})r_{k+2}^2, -\delta^2 - (70 + \frac{1}{2})r_{k+2}^2 \right) \right) \\ &\supset O^c \cap (B_{r_{k+3}}(0) \times (-73r_{k+2}^2, -71r_{k+2}^2)) \\ &= O^c \cap (B_{r_{k+3}}(0) \times (-1168r_{k+3}^2, -1136r_{k+3}^2)). \end{aligned}$$

Finally,

$$\begin{aligned} &\inf_{(y_0, s_0) \in (B_\delta(0) \times (-\delta^2, \delta^2)) \cap O} u_1(y_0, s_0) \\ &\gtrsim -1 + \int_{r_{k_\delta+3}}^1 \frac{\text{Cap}_{2,1,q'}(O^c \cap (B_\rho(0) \times (-17b\rho^2, -b\rho^2)))}{\rho^N} \frac{d\rho}{\rho} \quad \text{with } b = 1136 \\ &\gtrsim -1 + \int_{30r_{k_\delta+3}}^1 \frac{\text{Cap}_{2,1,q'}(O^c \cap (B_{\frac{\rho}{30}}(0) \times (-30\rho^2, -\rho^2)))}{\rho^N} \frac{d\rho}{\rho} \rightarrow \infty \quad \text{as } \delta \rightarrow 0, \end{aligned}$$

and

$$\begin{aligned} &\inf_{(y_0, s_0) \in (B_\delta(0) \times (-\delta^2, \delta^2)) \cap O} u_2(y_0, s_0) \\ &\gtrsim -1 + \int_{30r_{k_\delta+3}}^1 \frac{\mathcal{PH}_1^N(O^c \cap (B_{\frac{\rho}{30}}(0) \times (-30\rho^2, -\rho^2)))}{\rho^N} \frac{d\rho}{\rho} \rightarrow \infty \quad \text{as } \delta \rightarrow 0. \end{aligned}$$

This completes the proof of Theorem 1.1-(i) and Theorem 1.2.

Remark 4.1 (Uniqueness) In [16], Marcus and Véron prove that condition (1.8) is not only a necessary and sufficient condition for the existence of a large solution to (1.7), but it

implies the uniqueness of a such a large solution when it is fulfilled. The main step for this proof is to show that there exists a constant $c = c(\Omega, q > 0)$ such that any couple of large solutions (u, \hat{u}) satisfies

$$u(x) \leq \hat{u}(x) \quad \forall x \in \Omega. \quad (4.25)$$

The above estimate which is the key stone for proving uniqueness cannot be obtained in the case of the parabolic equation (1.13) since the necessary condition and the sufficient condition in Theorem 1.1 do not complement completely.

4.3 The viscous Hamilton-Jacobi parabolic equations

In this section we apply our previous result to the question of existence of a large solution of the following type of parabolic viscous Hamilton-Jacobi equation

$$\begin{aligned} \partial_t u - \Delta u + a|\nabla u|^p + bu^q &= 0 & \text{in } O, \\ u &= \infty & \text{on } \partial_p O, \end{aligned} \quad (4.26)$$

where $a > 0, b > 0$ and $1 < p \leq 2, q \geq 1$. First, we show that such a large solution to (4.26) does not exist when $q = 1$. Equivalently, there is no function $u \in C^{2,1}(O)$ satisfying

$$\begin{aligned} \partial_t u - \Delta u + a|\nabla u|^p &\geq -bu & \text{in } O, \\ u &= \infty & \text{on } \partial_p O. \end{aligned} \quad (4.27)$$

for $a > 0, b > 0$ and $p > 1$. Indeed, assuming that such a function $u \in C^{2,1}(O)$ exists, we define

$$U(x, t) = u(x, t)e^{bt} - \frac{\varepsilon}{2}|x|^2,$$

for $\varepsilon > 0$ and denote by $(x_0, t_0) \in O \setminus \partial_p O$ the point where U achieves its minimum in O , i.e. $U(x_0, t_0) = \inf\{U(x, t) : (x, t) \in O\}$. Clearly, we have

$$\partial_t U(x_0, t_0) \leq 0, \quad \Delta U(x_0, t_0) \geq 0 \quad \text{and} \quad \nabla U(x_0, t_0) = 0.$$

Thus,

$$\partial_t u(x_0, t_0) \leq -bu(x_0, t_0), \quad -\Delta u(x_0, t_0) \leq -\varepsilon N e^{-bt_0} \quad \text{and} \quad a|\nabla u(x_0, t_0)|^p = a\varepsilon^p |x_0|^p e^{-pbt_0},$$

from which follows

$$\begin{aligned} \partial_t u(x_0, t_0) - \Delta u(x_0, t_0) + a|\nabla u(x_0, t_0)|^p &\leq -bu(x_0, t_0) + \varepsilon e^{-bt_0} \left(-N + a\varepsilon^{p-1} |x_0|^p e^{-(p-1)bt_0} \right) \\ &< -bu(x_0, t_0) \end{aligned}$$

for ε small enough, which is a contradiction.

Proof of Theorem 1.3. By Remark 3.3, we have

$$\inf\{v(x, t); (x, t) \in O\} \geq (q_1 - 1)^{-\frac{1}{q_1-1}} R^{-\frac{2}{q_1-1}}.$$

Take $V = \lambda v^{\frac{1}{\alpha}} \in C^{2,1}(O)$ for $\lambda > 0$. Thus $v = \lambda^{-\alpha} V^\alpha$,

$$\inf\{V(x, t); (x, t) \in O\} > 0 \geq \lambda(q_1 - 1)^{-\frac{1}{\alpha(q_1-1)}} R^{-\frac{2}{\alpha(q_1-1)}},$$

and

$$\partial_t v - \Delta v + v^{q_1} = \alpha \lambda^{-\alpha} V^{\alpha-1} \partial_t V - \alpha \lambda^{-\alpha} V^{\alpha-1} \Delta V + \alpha(1 - \alpha) \lambda^{-\alpha} V^{\alpha-1} \frac{|\nabla V|^2}{V} + \lambda^{-\alpha q_1} V^{\alpha q_1}.$$

This leads to

$$\partial_t V - \Delta V + (1 - \alpha) \frac{|\nabla V|^2}{V} + \alpha^{-1} \lambda^{-\alpha(q_1-1)} V^{\alpha q_1 - \alpha + 1} = 0 \quad \text{in } O.$$

Using Hölder's inequality we obtain

$$\begin{aligned} (1-\alpha)\frac{|\nabla V|^2}{V} + (2\alpha)^{-1}\lambda^{-\alpha(q_1-1)}V^{\alpha q_1-\alpha+1} &\geq c_1|\nabla V|^p\lambda^{-\frac{\alpha(q_1-1)(2-p)}{2}}V^{\frac{\alpha(q_1-1)(2-p)}{2}-(p-1)} \\ &\geq c_2|\nabla V|^p\lambda^{-(p-1)}R^{-2+p+\frac{2(p-1)}{\alpha(q_1-1)}}, \end{aligned}$$

and

$$(2\alpha)^{-1}\lambda^{-\alpha(q_1-1)}V^{\alpha q_1-\alpha+1} \geq c_3\lambda^{-(q-1)}R^{-2+\frac{2(q-1)}{\alpha(q_1-1)}}V^q.$$

If we choose

$$\lambda = \min\{c_2^{\frac{1}{p-1}}, c_3^{\frac{1}{q-1}}\} \min\left\{a^{-\frac{1}{p-1}}R^{-\frac{2-p}{p-1}+\frac{2}{\alpha(q_1-1)}}, b^{-\frac{1}{q-1}}R^{-\frac{2}{q-1}+\frac{2}{\alpha(q_1-1)}}\right\},$$

then

$$\begin{aligned} c_2\lambda^{-(p-1)}R^{-2+p+\frac{2(p-1)}{\alpha(q_1-1)}} &\geq a, \\ c_3\lambda^{-(q-1)}R^{-2+\frac{2(q-1)}{\alpha(q_1-1)}} &\geq b, \end{aligned}$$

from what follows

$$\partial_t V - \Delta V + a|\nabla V|^p + bV^q \leq 0 \quad \text{in } O.$$

By Remark 3.5, there exists a maximal solution $u \in C^{2,1}(O)$ of

$$\partial_t u - \Delta u + a|\nabla u|^p + bu^q = 0 \quad \text{in } O.$$

Therefore, $u \geq V = \lambda v^{\frac{1}{\alpha}}$ and u is a large solution of (4.26). This completes the proof of Theorem 1.3. \blacksquare

5 Appendix

Proof of Proposition 2.5.

Step 1. We claim that the following relation holds:

$$\int_{\mathbb{R}^{N+1}} (\mathbb{I}_2^1[\mu](x, t))^{(N+2)/N} dx dt \asymp \int_{\mathbb{R}^{N+1}} \int_0^1 (\mu(\tilde{Q}_r(x, t)))^{2/N} \frac{dr}{r} d\mu(x, t). \quad (5.1)$$

In fact, we have for $\rho_j = 2^{-j}$, $j \in \mathbb{Z}$,

$$\begin{aligned} \sum_{j=1}^{\infty} \int_{\mathbb{R}^{N+1}} (\mu(\tilde{Q}_{\rho_j}(x, t)))^{2/N} d\mu(x, t) &\lesssim \int_{\mathbb{R}^{N+1}} \int_0^1 (\mu(\tilde{Q}_r(x, t)))^{2/N} \frac{dr}{r} d\mu(x, t) \\ &\lesssim \sum_{j=0}^{\infty} \int_{\mathbb{R}^{N+1}} (\mu(\tilde{Q}_{\rho_j}(x, t)))^{2/N} d\mu(x, t). \end{aligned}$$

Note that for any $j \in \mathbb{Z}$

$$\begin{aligned} \rho_j^{-N-2} \int_{\mathbb{R}^{N+1}} (\mu(\tilde{Q}_{\rho_{j+1}}(x, t)))^{(N+2)/N} dx dt &\lesssim \int_{\mathbb{R}^{N+1}} (\mu(\tilde{Q}_{\rho_j}(x, t)))^{2/N} d\mu(x, t) \\ &\lesssim \rho_j^{-N-2} \int_{\mathbb{R}^{N+1}} (\mu(\tilde{Q}_{\rho_{j-1}}(x, t)))^{(N+2)/N} dx dt. \end{aligned}$$

Thus,

$$\begin{aligned} \sum_{j=2}^{\infty} \rho_j^{-N} \int_{\mathbb{R}^{N+1}} (\mu(\tilde{Q}_{\rho_j}(x, t)))^{(N+2)/N} dx dt &\lesssim \int_{\mathbb{R}^{N+1}} \int_0^1 (\mu(\tilde{Q}_r(x, t)))^{2/N} \frac{dr}{r} d\mu(x, t) \\ &\lesssim \sum_{j=-1}^{\infty} \rho_j^{-N} \int_{\mathbb{R}^{N+1}} (\mu(\tilde{Q}_{\rho_j}(x, t)))^{(N+2)/N} dx dt. \end{aligned}$$

This yields

$$\begin{aligned} \int_{\mathbb{R}^{N+1}} \left(\mathbb{M}_2^{1/4}[\mu](x, t) \right)^{(N+2)/N} dx dt &\lesssim \int_{\mathbb{R}^{N+1}} \int_0^1 (\mu(\tilde{Q}_r(x, t)))^{2/N} \frac{dr}{r} d\mu(x, t) \\ &\lesssim \int_{\mathbb{R}^{N+1}} \left(\mathbb{I}_2^4[\mu](x, t) \right)^{(N+2)/N} dx dt. \end{aligned}$$

By [21, Theorem 4.2],

$$\int_{\mathbb{R}^{N+1}} \left(\mathbb{M}_2^{1/4}[\mu](x, t) \right)^{(N+2)/N} dx dt \asymp \int_{\mathbb{R}^{N+1}} \left(\mathbb{I}_2^4[\mu](x, t) \right)^{(N+2)/N} dx dt,$$

thus we obtain (5.1).

Step 2. End of the proof. The first inequality in (2.1) is proved in [21]. We now prove the second inequality. By Theorem 2.4 there is $\mu \in \mathfrak{M}^+(\mathbb{R}^{N+1})$, $\text{supp}(\mu) \subset K$ such that

$$\|\mathbb{M}_2^2[\mu]\|_{L^\infty(\mathbb{R}^{N+1})} \leq 1 \quad \text{and} \quad \mu(K) \asymp \mathcal{P}\mathcal{H}_2^N(K) \gtrsim |K|^{N/(N+2)}. \quad (5.2)$$

Thanks to (5.1), we have for $\delta = \min\{1, (\mu(K))^{1/N}\}$

$$\begin{aligned} \|\mathbb{I}_2^1[\mu]\|_{L^{(N+2)/N}(\mathbb{R}^{N+1})}^{(N+2)/N} &\asymp \int_{\mathbb{R}^{N+1}} \int_0^1 (\mu(\tilde{Q}_r(x, t)))^{2/N} \frac{dr}{r} d\mu(x, t) \\ &\asymp \int_{\mathbb{R}^{N+1}} \left(\int_0^\delta + \int_\delta^1 \right) (\mu(\tilde{Q}_r(x, t)))^{2/N} \frac{dr}{r} d\mu(x, t) \\ &\lesssim \int_0^\delta r^2 \frac{dr}{r} \int_{\mathbb{R}^{N+1}} d\mu(x, t) + \int_\delta^1 \frac{dr}{r} \left(\int_{\mathbb{R}^{N+1}} d\mu(x, t) \right)^{(N+2)/N} \\ &\lesssim (\mu(K))^{(N+2)/N} (1 + \log_+((\mu(K))^{-1})) \\ &\lesssim (\mu(K))^{(N+2)/N} \log \left(\frac{|\tilde{Q}_{200}(0, 0)|}{|K|} \right). \end{aligned}$$

Set $\tilde{\mu} = \left(\log \left(\frac{|\tilde{Q}_{200}(0, 0)|}{|K|} \right) \right)^{-N/(N+2)} \mu / \mu(K)$, then $\|\mathbb{I}_2^1[\tilde{\mu}]\|_{L^{(N+2)/N}(\mathbb{R}^{N+1})} \lesssim 1$.

It is well known that

$$\text{Cap}_{2,1, \frac{N+2}{2}}(K) \asymp \sup\{(\omega(K))^{(N+2)/2} : \omega \in \mathfrak{M}^+(K), \|\mathbb{I}_2^1[\omega]\|_{L^{(N+2)/N}(\mathbb{R}^{N+1})} \lesssim 1\} \quad (5.3)$$

see [21, Section 4]. This gives the second inequality in (2.1).

It is easy to prove (2.2) from its definition. Moreover, (5.3) implies that

$$\frac{1}{\text{Cap}_{2,1, \frac{N+2}{2}}(K)^{2/N}} \asymp \inf\{\|\mathbb{I}_2^1[\omega]\|_{L^{(N+2)/N}(\mathbb{R}^{N+1})}^{(N+2)/N} : \omega \in \mathfrak{M}^+(K), \omega(K) = 1\}.$$

We deduce from (5.1) that

$$\frac{1}{\text{Cap}_{2,1, \frac{N+2}{2}}(K)^{2/N}} \asymp \inf \left\{ \int_{\mathbb{R}^{N+1}} \int_0^1 (\omega(\tilde{Q}_r(x, t)))^{2/N} \frac{dr}{r} d\mu(x, t) : \omega \in \mathfrak{M}^+(K), \omega(K) = 1 \right\}. \quad (5.4)$$

As in [12, proof of Lemma 2.2], it is easy to derive (2.3) from (5.4). \blacksquare

Proof of Proposition 2.6. Thanks to the Poincaré inequality, it is enough to show that there exists $\varphi \in C_c^\infty(\tilde{Q}_{3/2}(0, 0))$ such that $0 \leq \varphi \leq 1$, with $\varphi = 1$ in an open neighborhood of K and

$$\int_{\mathbb{R}^{N+1}} (|D^2\varphi|^p + |\partial_t\varphi|^p) dxdt \lesssim \text{Cap}_{2,1,p}(K). \quad (5.5)$$

By definition, one can find $0 \leq \phi \in S(\mathbb{R}^{N+1})$, $\phi \geq 1$ in a neighborhood of K such that

$$\int_{\mathbb{R}^{N+1}} (|D^2\phi|^p + |\nabla\phi|^p + |\phi|^p + |\partial_t\phi|^p) dxdt \leq 2\text{Cap}_{2,1,p}(K).$$

Let η be a cut off function on $\tilde{Q}_1(0, 0)$ with respect to $\tilde{Q}_{3/2}(0, 0)$ and $H \in C^\infty(\mathbb{R})$ such that

$$0 \leq H(t) \leq t^+, \quad |t| |H''(t)| \lesssim 1 \quad \text{for all } t \in \mathbb{R}, \quad H(t) = 0 \quad \text{for } t \leq 1/4 \quad \text{and} \quad H(t) = 1 \quad \text{for } t \geq 3/4.$$

We claim that

$$\int_{\mathbb{R}^{N+1}} (|D^2\varphi|^p + |\partial_t\varphi|^p) dxdt \lesssim \int_{\mathbb{R}^{N+1}} (|D^2\phi|^p + |\nabla\phi|^p + |\phi|^p + |\partial_t\phi|^p) dxdt, \quad (5.6)$$

where $\varphi = \eta H(\phi)$. Indeed, we have

$$|D^2\varphi| \lesssim |D^2\eta|H(\phi) + |\nabla\eta||H'(\phi)||\nabla\phi| + \eta|H''(\phi)||\nabla\phi|^2 + \eta|H'(\phi)||D^2\phi|,$$

and

$$|\partial_t\varphi| \lesssim |\partial_t\eta|H(\phi) + \eta|H'(\phi)||\phi_t|, \quad H(\phi) \leq \phi, \quad \phi|H''(\phi)| \lesssim 1.$$

Thus,

$$\begin{aligned} \int_{\mathbb{R}^{N+1}} (|D^2\varphi|^p + |\partial_t\varphi|^p) dxdt &\lesssim \int_{\mathbb{R}^{N+1}} (|D^2\phi|^p + |\nabla\phi|^p + |\phi|^p + |\partial_t\phi|^p) dxdt \\ &\quad + \int_{\mathbb{R}^{N+1}} \frac{|\nabla\phi|^{2p}}{\phi^p} dxdt. \end{aligned}$$

This implies (5.6) since, according to [1], one has

$$\int_{\mathbb{R}^N} \frac{|\nabla\phi(t)|^{2p}}{(\phi(t))^p} dx \lesssim \int_{\mathbb{R}^N} |D^2\phi(t)|^p dx \quad \forall t \in \mathbb{R}. \quad \blacksquare$$

References

- [1] D. R. Adams. *On the existence of capacitary strong type estimates in \mathbb{R}^N* , Ark. Mat. **14**, 125-140 (1976).
- [2] R.J. Bagby. *Lebesgue spaces of parabolic potentials*, Ill. J. Math. **15**, 610-634 (1971).
- [3] P. Baras and M. Pierre. *Problèmes paraboliques semi-linéaires avec données mesures*, Applicable Anal. **18**, 111-149 (1984).
- [4] P. Baras, M. Pierre, *Critère d'existence des solutions positives pour des équations semi-linéaires non monotones*, Ann. Inst. H. Poincaré, Anal. Non Lin. **3**, (1985), 185-212.
- [5] J. S. Dthersin and J. F. Le Gall. *Wieners test for super-Brownian motion and the Brownian snake*, Probab. Theory Relat. Fields **108**, 103-29 (1997).

- [6] E. DiBenedetto. *Degenerate parabolic equations*. Universitext. Springer-Verlag, New York, (1993).
- [7] E. B. Dynkin, S. E. Kuznetsov, *Superdiffusions and removable singularities for quasilinear partial differential equations*. Comm. Pure Appl. Math. **49**, 125-176 (1996).
- [8] E. B. Dynkin, S. E. Kuznetsov, *Fine topology and fine trace on the boundary associated with a class of semilinear differential equations*. Comm. Pure Appl. Math. **51**, 897-936 (1998).
- [9] L. C. Evans, R. F. Gariepy, *Wiener criterion for the heat equation*. Arch. Rat. Mech. Anal. **78**, 293-314 (1982).
- [10] N. Garofalo and E. Lanconelli, *Wiener's criterion for parabolic equations with variable coefficients and its consequences*, Trans. Amer. Math. Soc. **308**, 811-836 (1988).
- [11] R. F. Gariepy and W.P. Ziemer, *Thermal capacity and boundary regularity*, J. Diff. Eq. **45**, 374-388 (1982).
- [12] D. Labutin. *Wiener regularity for large solutions of nonlinear equations*, Archiv for Math. **41**, 307-339 (2003).
- [13] J.-F. Le Gall, *Branching processes, random trees and superprocesses*. Proceedings of the International Congress of Mathematicians (Berlin, 1998), Doc. Math., Extra Vol. III 279-289 (1998).
- [14] J.-F. Le Gall, *Spatial branching processes, random snakes and partial differential equations*. Lectures in Mathematics ETH Zürich. Birkhäuser Verlag, Basel, 1999.
- [15] G.M. Lieberman, *Second Order Parabolic Differential Equations*, World Scientific press, River Edge (1996).
- [16] M. Marcus and L. Véron, *Maximal solutions for $-\Delta u + u^q = 0$ in open and finely open sets*, J. Math. Pures Appl. **91** 256-295 (2009).
- [17] M. Marcus and L. Véron, *Nonlinear second order elliptic equations involving measures*, Series in Nonlinear Analysis and Applications **21**, De Gruyter, Berlin/Boston (2013).
- [18] M. Marcus and L. Véron. *Capacitary estimates of solutions of semilinear parabolic equations*, Calc. Var. & Part. Diff. Eq. **48**, 131-183 (2013).
- [19] T. Nguyen-Phuoc *Parabolic equations with exponential nonlinearity and measure data*, J. Diff. Eq. **257**, 2704-2727 (2014).
- [20] Quoc-Hung Nguyen and L. Véron., *Wiener criteria for existence of large solutions of quasilinear elliptic equations with absorption*, Potential Analysis **42**, 681-697 (2015)
- [21] Quoc-Hung Nguyen, *Potential estimates and quasilinear equations with measure data* arXiv:1405.2587v1.
- [22] B. O. Tureson. *Nonlinear Potential Theory and weighted Sobolev Spaces*, Lecture Notes in Mathematics, Vol. 1736, Springer-Verlag (2000).
- [23] L. Véron. *A note on maximal solutions of nonlinear parabolic equations with absorption*, Asymptotic Anal. **72**, 189-200 (2011).
- [24] W.P. Ziemer, *Behavior at the boundary of solution of quasilinear parabolic equations*, J. Diff. Eq. **35**, 291-305 (1980).